

The orbifold-string theories of permutation-type: II. Cycle dynamics and target space-time dimensions

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Abstract

We continue our discussion of the general bosonic prototype of the new orbifold-string theories of permutation-type. Supplementing the extended physical-state conditions of the previous paper, we construct here the extended Virasoro generators with cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$, where $f_j(\sigma)$ is the length of cycle j in twisted sector σ . We also find an equivalent, reduced formulation of each physical-state problem at reduced cycle central charge $c_j(\sigma) = 26$. These tools are used to begin the study of the target space-time dimension $\hat{D}_j(\sigma)$ of cycle j in sector σ , which is naturally defined as the number of zero modes (momenta) of each cycle. The general model-dependent formulae derived here will be used extensively in succeeding papers, but are evaluated in this paper only for the simplest case of the “pure” permutation orbifolds.

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1 Introduction

The new orbifold-string theories of permutation-type [1-6] include the bosonic prototypes

$$\frac{U(1)^{26K}}{H_+} = \frac{U(1)_1^{26} \times \cdots \times U(1)_K^{26}}{H_+}, \quad H_+ \subset H(\text{perm})_K \times H'_{26}, \quad (1.1a)$$

$$\left[\frac{U(1)^{26K}}{H_+} \right]_{\text{open}}, \quad (1.1b)$$

$$\frac{U(1)^{26K}}{H_-} = \frac{U(1)_L^{26} \times U(1)_R^{26}}{H_-}, \quad H_- \subset \mathbb{Z}_2(\text{w.s.}) \times H'_{26} \quad (1.1c)$$

and generalizations of these, as noted in Appendix B of Ref. 6. The three families in (1.1) are called respectively the generalized permutation orbifolds (twisted closed strings at sector central charge $\hat{c} = 26K$), the open-string analogues of the generalized permutation orbifolds (twisted open strings at $\hat{c} = 26K$), and the orientation-orbifold string systems, which contain an equal number of twisted closed strings at $\hat{c} = 26$ and twisted open strings at $\hat{c} = 52$. The open-string sectors of the orientation orbifolds are contained, along with their T -duals, at $K = 2$ in the open-string analogues of the generalized permutation orbifolds. The closed-string sectors of the orientation orbifolds form the ordinary space-time orbifold $U(1)^{26}/H'_{26}$ at $\hat{c} = 26$. Further information on special cases of these orbifold-string systems, especially $\hat{c} = 52$, is contained in Refs. [3-5]. We note in particular that the orientation-orbifold string systems (1.1c) generalize and include [4] the ordinary critical bosonic open-closed string system.

In the previous paper [6] of the present series, cycle-bases of general permutation groups and the principles of the orbifold program [7-21] were used to construct a twisted BRST system for each cycle j in each twisted sector σ of these orbifolds, including the extended algebra of the BRST charges

$$[\hat{Q}_i(\sigma), \hat{Q}_j(\sigma)]_+ = 0 \quad \forall i, j \text{ in sector } \sigma, \quad (1.2)$$

and right-mover copies of these systems in the twisted closed-string sectors. Moreover, the BRST systems were used to find the *extended physical-state conditions* of the matter in

cycle j of sector σ

$$(\hat{L}_{jj}((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) - \hat{a}_{f_j(\sigma)} \delta_{m + \frac{\hat{j}}{f_j(\sigma)}, 0}) |\chi(\sigma)\rangle_j = 0, \quad (1.3a)$$

$$\begin{aligned} & [\hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{L}_{\ell\ell}(m + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ &= \delta_{j\ell} \left\{ (m - n - \frac{\hat{j} - \hat{\ell}}{f_j(\sigma)}) \hat{L}_{\hat{j} + \hat{\ell}, j}(m + n + \frac{\hat{j} + \hat{\ell}}{f_j(\sigma)}) \right. \\ & \quad \left. + \frac{1}{12} \hat{c}_j(\sigma) (m + \frac{\hat{j}}{f_j(\sigma)}) ((m + \frac{\hat{j}}{f_j(\sigma)})^2 - 1) \delta_{m + n + \frac{\hat{j} + \hat{\ell}}{f_j(\sigma)}, 0} \right\}, \end{aligned} \quad (1.3b)$$

$$\hat{c}_j(\sigma) = 26f_j(\sigma), \quad \hat{a}_{f_j(\sigma)} = \frac{13f_j^2(\sigma) - 1}{12f_j(\sigma)}, \quad (1.3c)$$

$$\bar{j} = 0, 1, \dots, f_j(\sigma) - 1, \quad j = 0, 1, \dots, N(\sigma) - 1, \quad \sum_j f_j(\sigma) = K \quad (1.3d)$$

including again a right-mover copy of these conditions for twisted closed-string sectors. The algebra (1.3b) of the matter generators $\{\hat{L}_{jj}\}$ is called the *orbifold Virasoro algebra* [7,15,6] of cycle j in sector σ . The fundamental numbers (1.3c) of each cycle are the *cycle central charge* $\hat{c}_j(\sigma)$ and the *cycle-intercept* $\hat{a}_{f_j(\sigma)}$, both expressed in terms of the length $f_j(\sigma)$ of cycle j in sector σ . Using the final sum rule in Eq. (1.3d) the reader easily verifies that the *sector central charges*

$$\hat{c}(\sigma) = \sum_j \hat{c}_j(\sigma) \quad (1.4)$$

are $26K$ for the closed- and open-string counterparts of the generalized permutation orbifolds and 52 ($K = f_0(\sigma) = 2$) for the twisted open-string sectors of the orientation orbifolds. The twisted closed-string sectors of the orientation orbifolds (the ordinary space-time orbifold $U(1)^{26}/H'_{26}$) can also be obtained from these results by choosing $K = N(\sigma) = f_0(\sigma) = 1$ and hence the ordinary values $\hat{a}_1 = 1, \hat{c}(\sigma) = \hat{c}_0(\sigma) = 26$.

In fact these results see only the permutation subgroup $H(\text{perm})_K$ or $\mathbb{Z}_2(\text{w.s.})$ of H_\pm , which determines the twisted permutation gravities [2] of each sector and hence the BRST systems. The 26-dimensional automorphism subgroup H'_{26} of H_\pm , which operates uniformly on each left- and right-mover copy of the critical closed string, is encoded however in the explicit form of the extended Virasoro generators of the matter.

Our first task in this paper is therefore to supplement the extended physical-state conditions (1.3) with the construction of the orbifold Virasoro generators $\{\hat{L}_{jj}\}$ at cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$ as functions of the twisted matter. This construction encodes the solution of the spectral problem of each element $\omega(\sigma) \in H'_{26}$ of the 26-dimensional automorphism subgroup, and our general formulae will be evaluated explicitly for a large class of examples of H'_{26} in the following paper. Subexamples of this construction at $\hat{c}(\sigma) = 52$ and $\hat{c}(\sigma) = 26\lambda, \lambda$ prime have already been discussed in Refs. [3-5].

Generalizing our work at $\hat{c}(\sigma) = 52$ in Ref. [3], we shall also find an equivalent, *reduced* form of the physical-state problem for each cycle j of each sector σ at *reduced* cycle central charge $c_j(\sigma) = 26$.

With these tools, we shall begin a survey of the *space-time (target-space) interpretation* of the orbifold-string theories, noting in particular with Ref. [3] that the target space-times are invariant under the reduction. In this discussion, we will focus on the *target space-time dimension*

$$\hat{D}_j(\sigma) \equiv \dim\{\hat{J}_j(0)_\sigma\} \quad (1.5)$$

of cycle j in sector σ as the number of zero modes (momenta) of the cycle, and following Refs. [3-5], we will define the momentum-squared operators and level-spacing which are needed to analyze the extended physical-state problems. In the only explicit examples of this paper, we shall find that $\hat{D}_j(\sigma) = 26$ for the “pure” permutation orbifolds (with trivial H'_{26}), so that these simple cases are equivalent to collections of ordinary critical strings. More generally however *the dimensionality of the target space-time is not equal to any of the central charges of the theories*, and in succeeding papers we will present many examples of non-trivial H'_{26} with $\hat{D}_j(\sigma) \leq 26$!

2 An application of the orbifold program

To obtain the general forms of the orbifold Virasoro generators for each cycle j of each sector σ , we apply the standard methods of the orbifold program [7-21] which emphasizes the principle of local isomorphisms [9,11,12,15-17].

The orbifold program always begins with the operator-product formulation of the untwisted systems in question. Here we need then only the operator-product form of the stress-tensor/current system of K copies of the critical bosonic string:

$$T_I(z) = \frac{1}{2}G^{ab} :J_{aI}(z)J_{bI}(z):, \quad I = 0, 1, \dots, K-1, \quad (2.1a)$$

$$G = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad a, b = 0, 1, \dots, 25, \quad (2.1b)$$

$$\begin{aligned} T_I(z)T_J(w) = \delta_{IJ} \left(\frac{26/2}{(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) T_I(w) \right) \\ + :T_I(z)T_J(w): \end{aligned} \quad (2.1c)$$

$$\begin{aligned} T_I(z)J_{aJ}(w) = \delta_{IJ} \left(\frac{1}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) J_{aI}(w) \\ + :T_I(z)J_{aJ}(w): \end{aligned} \quad (2.1d)$$

$$J_{aI}(z)J_{bJ}(w) = \frac{\delta_{IJ}G_{ab}}{(z-w)^2} + :J_{aI}(z)J_{bJ}(w):. \quad (2.1e)$$

The symbol $:\cdots:$ is operator-product normal ordering, that is, the operator product minus the singularities shown. The total stress tensor $T(z) = \sum_I T_I(z)$ is also Virasoro with total central charge $c = 26K$, and a right-mover copy of this system is included implicitly in our application below.

The automorphic responses of these operators are

$$J_{aI}(z)' = \omega(\sigma)_a^b W(\sigma)_I^J J_{bJ}(z), \quad (2.2a)$$

$$T_I(z)' = W(\sigma)_I^J T_J(z), \quad (2.2b)$$

$$W(\sigma) \in H(\text{perm})_K, \quad \omega(\sigma) \in H'_{26}. \quad (2.2c)$$

Note that the definition of a sector σ requires the choice of one element $(W \times \omega)$ from each equivalence class of both $H(\text{perm})_K$ and H'_{26} . Here we are following the sector-labeling convention of the orbifold program for product groups, but we mention that σ can equivalently be viewed as a two-component vector, with one component each for $H(\text{perm})_K$ and H'_{26} .

The next step in the orbifold program is to find the so-called eigenfields [9,11,12,15-17] under the automorphism groups, and for this we must first recall the *H-eigenvalue problems* of the group elements: For each element $W(\sigma) \in H(\text{perm})_K$ we have the spectral problem

$$W(\sigma)_I^J V^\dagger(\sigma)_J^{\hat{j}j} = V^\dagger(\sigma)_I^{\hat{j}j} e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}}, \quad (2.3a)$$

$$\bar{j} = 0, 1, \dots, f_j(\sigma) - 1, \quad j = 0, 1, \dots, N(\sigma) - 1, \quad (2.3b)$$

$$\sum_j = N(\sigma), \quad \sum_j f_j(\sigma) = K, \quad (2.3c)$$

where j labels cycles of length $f_j(\sigma)$, $N(\sigma)$ is the number of cycles in $W(\sigma)$, and \hat{j} indexes within each cycle j . The explicit form of the unitary eigenmatrix $V(\sigma)$ is given in Refs. [13,15], and this eigenvalue problem was also discussed in the previous paper [6] on the general BRST problem. What is essential to add here is the eigenvalue problem for each element $\omega(\sigma) \in H'_{26}$ of the 26-dimensional automorphism subgroup:

$$\omega(\sigma)_a^b U^\dagger(\sigma)_b^{n(r)\mu} = U^\dagger(\sigma)_a^{n(r)\mu} e^{-2\pi i \frac{n(r)}{\rho(\sigma)}} \quad (2.4a)$$

$$\bar{n}(r) \in \{0, 1, \dots, \rho(\sigma) - 1\}, \quad \sum_\mu = \dim[\bar{n}(r)], \quad \sum_r \dim[\bar{n}(r)] = 26. \quad (2.4b)$$

Here $U(\sigma)$ is the unitary eigenmatrix of $\omega(\sigma)$, with order $\rho(\sigma)$, and $n(r)$, $\mu = \mu(n(r))$ are respectively the spectral and degeneracy indices of $\omega(\sigma)$. The barred quantities in (2.3b) and (2.4b) are the pullbacks of the spectral indices to their fundamental ranges. Many of these spectral problems [9,11,13,15,16] have been solved explicitly in the orbifold program, but we will not choose any particular non-trivial H'_{26} in the general discussion of this paper (see however Sec. 10).

Given the forms of these two eigenvalue problems, we may write down the *eigenfields* for each $W(\sigma) \in H(\text{perm})_K$ and $\omega(\sigma) \in H'_{26}$:

$$\Theta_{\hat{j}j}(z, \sigma) \equiv \sqrt{f_j(\sigma)} V(\sigma)_{\hat{j}j}^I T_I(z), \quad (2.5a)$$

$$\mathcal{J}_{n(r)\mu\hat{j}j}(z, \sigma) \equiv \chi_{n(r)\mu}(\sigma) \sqrt{f_j(\sigma)} U(\sigma)_{n(r)\mu}^a V(\sigma)_{\hat{j}j}^I J_{aI}(z). \quad (2.5b)$$

Here we have chosen the standard normalization $\chi_{\hat{j}j}(\sigma) = \sqrt{f_j(\sigma)}$ for elements of $H(\text{perm})_K$, but left the normalizations $\chi_{n(r)\mu}(\sigma)$ arbitrary for elements of H'_{26} . The eigenfields are constructed to diagonalize the automorphic responses as follows:

$$\Theta_{\hat{j}j}(z, \sigma)' = e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}} \Theta_{\hat{j}j}(z, \sigma), \quad (2.6a)$$

$$\mathcal{J}_{n(r)\mu\hat{j}j}(z, \sigma)' = e^{-2\pi i (\frac{\hat{j}}{f_j(\sigma)} + \frac{n(r)}{\rho(\sigma)})} \mathcal{J}_{n(r)\mu\hat{j}j}(z, \sigma). \quad (2.6b)$$

Moreover, the eigenfields inherit the following periodicity conditions

$$\Theta_{\hat{j} \pm f_j(\sigma), j}(z, \sigma) = \theta_{\hat{j}j}(z, \sigma), \quad (2.7a)$$

$$\mathcal{J}_{n(r) \pm \rho(\sigma), \mu \hat{j}j}(z, \sigma) = \mathcal{J}_{n(r)\mu, \hat{j} \pm f_j(\sigma), j}(z, \sigma) = \mathcal{J}_{n(r)\mu\hat{j}j}(z, \sigma) \quad (2.7b)$$

from the natural periodicities of the eigenvalue problems.

The composite form and operator products of the eigenfields in terms of themselves are then straightforwardly computed from their definitions and the original operator products (2.1). We will not write them out explicitly here (see however the remark after Eq. (3.4)), but call attention only to some useful quantities, the *twisted metrics*, which appear in the operator products of the eigenfields:

$$\begin{aligned} \mathcal{G}_{\hat{j}j; \hat{\ell}\ell}(\sigma) &= \sqrt{f_j(\sigma)} \sqrt{f_\ell(\sigma)} V(\sigma)_{\hat{j}j}^I V(\sigma)_{\hat{\ell}\ell}^J \delta_{IJ} \\ &= \delta_{j\ell} f_j(\sigma) \delta_{\hat{j} + \hat{\ell}, 0 \bmod f_j(\sigma)}, \end{aligned} \quad (2.8a)$$

$$\begin{aligned} \mathcal{G}_{n(r)\mu; n(s)\nu}(\sigma) &= \chi_{n(r)\mu}(\sigma) \chi_{n(s)\nu}(\sigma) U(\sigma)_{n(r)\mu}^a U(\sigma)_{n(s)\nu}^b G_{ab} \\ &= \delta_{n(r)+n(s), 0 \bmod f_j(\sigma)} \mathcal{G}_{n(r)\mu; -n(r)\nu}(\sigma) \end{aligned} \quad (2.8b)$$

$$\sum_{n(t), \eta} \mathcal{G}^{n(r)\mu; n(t)\eta}(\sigma) \mathcal{G}_{n(t)\eta; n(s)\nu}(\sigma) = \delta_{n(r)\mu}^{n(s)\nu} \quad (2.8c)$$

In what follows, the information about the choice of $\omega(\sigma) \in H'_{26}$ is encoded in the quantities $n(r)\mu$, $\rho(\sigma)$ and the twisted metric $\mathcal{G}(\sigma)$ and its inverse $\mathcal{G}^(\sigma)$ in Eqs. (2.8b,c). The inverse of the metric (2.8a) is obtained by inverting the factor $f_j(\sigma)$, while the inverse of Eq. (2.8b) involves the inverse of the normalizations and replacement of the eigenmatrices by their adjoints.

At this stage we have only rearranged the untwisted theory in terms of the eigenfields $\mathcal{A}(z, \sigma)$. The final step in the orbifold program is the transition to twisted sector σ of the orbifold by an application of the *principle of local isomorphisms* [9,11,12,15-17]

$$\mathcal{A}(z, \sigma) \rightarrow \hat{A}(z, \sigma), \quad (2.9a)$$

$$\text{operator products of } \{\mathcal{A}(z, \sigma)\} \rightarrow \text{operator products of } \{\hat{A}(z, \sigma)\}, \quad (2.9b)$$

$$\text{diagonal automorphic responses} \rightarrow \text{monodromies}, \quad (2.9c)$$

where $\{\hat{A}(z, \sigma)\}$ are now the twisted fields of twisted sector σ of the orbifold. The name of the principle derives from part (b) of Eq. (2.9), that the operator products of the twisted fields are the same as (locally isomorphic to) the operator products of the eigenfields. (We remind that there is another, equivalent way around the commuting diagrams of Refs. [9,11,17] to get from the untwisted fields to the twisted fields. This path involves first a parallel application of the principle of local isomorphisms, followed by a monodromy decomposition to obtain the twisted fields \hat{A} .)

3 The twisted operator products of sector σ

Having completed the steps above, we emerge in the orbifold with the following twisted stress tensors of sector σ

$$\hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) = \sum_{n(r)\mu\nu} \frac{\mathcal{G}^{n(r)\mu; -n(r)\nu}(\sigma)}{2f_j(\sigma)} \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} : \hat{J}_{n(r)\mu\hat{\ell}\hat{j}}(z, \sigma) \hat{J}_{-n(r), \nu, \hat{j}-\hat{\ell}, \hat{j}}(z, \sigma) : \quad (3.1)$$

where $: \cdots :$ is now operator-product normal ordering in the orbifold. The monodromies of these operators are

$$\hat{\theta}_{\hat{j}\hat{j}}(ze^{2\pi i}, \sigma) = e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}} \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma), \quad (3.2a)$$

$$\hat{J}_{n(r)\mu\hat{j}\hat{j}}(ze^{2\pi i}, \sigma) = e^{-2\pi i (\frac{\hat{j}}{f_j(\sigma)} + \frac{n(r)}{\rho(\sigma)})} \hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) \quad (3.2b)$$

and the periodicities

$$\hat{\theta}_{\hat{j}\pm f_j(\sigma), \hat{j}}(z, \sigma) = \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma), \quad (3.3a)$$

$$\hat{J}_{n(r)\pm \rho(\sigma), \mu\hat{j}\hat{j}}(z, \sigma) = \hat{J}_{n(r)\mu, \hat{j}\pm f_j(\sigma), \hat{j}}(z, \sigma) = \hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) \quad (3.3b)$$

are inherited from the eigenfields.

The operator products of sector σ are obtained as

$$\begin{aligned} \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) \hat{\theta}_{\hat{\ell}\hat{\ell}}(\omega, \sigma) &= \delta_{j\ell} \left[\frac{\delta_{\hat{j}+\hat{\ell}, 0 \bmod f_j(\sigma)} \frac{26}{2} f_j(\sigma)}{(z-\omega)^4} \right. \\ &\quad \left. + \left(\frac{2}{(z-\omega^2)^2} + \frac{1}{z-\omega} \partial_\omega \right) \hat{\theta}_{\hat{j}+\hat{\ell}, j}(\omega, \sigma) \right] \\ &\quad + : \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) \hat{\theta}_{\hat{\ell}\hat{\ell}}(\omega, \sigma) : \end{aligned} \quad (3.4a)$$

$$\begin{aligned} \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) \hat{J}_{n(r)\mu\hat{\ell}\hat{\ell}}(\omega, \sigma) &= \delta_{j\ell} \left(\frac{1}{(z-\omega)^2} + \frac{1}{z-\omega} \partial_\omega \right) \hat{J}_{n(r)\mu, \hat{j}+\hat{\ell}, j}(\omega, \sigma) \\ &\quad + : \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) \hat{J}_{n(r)\mu\hat{\ell}\hat{\ell}}(\omega, \sigma) : \end{aligned} \quad (3.4b)$$

$$\begin{aligned} \hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\hat{\ell}}(\omega, \sigma) &= \delta_{j\ell} \left(\frac{f_j(\sigma) \mathcal{G}_{n(r)\mu; n(s)\nu}(\sigma) \delta_{\hat{j}+\hat{\ell}, 0 \bmod f_j(\sigma)}}{(z-\omega)^2} \right) \\ &\quad + : \hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\hat{\ell}}(\omega, \sigma) : . \end{aligned} \quad (3.4c)$$

The relations above provide a complete description of twisted sector σ . If desired, the previously-omitted details of the eigenfield system can be obtained from these statements by going backward $\hat{A} \rightarrow \mathcal{A}$, while replacing the monodromies (3.2) with the diagonal automorphic responses (2.6) of the eigenfields. We will comment on applications to specific orbifold-string systems after finding the corresponding mode algebras below.

4 The twisted mode algebras of sector σ

The twisted operator-product form of the system above is straightforwardly translated to the mode-algebraic description of the sector. With attention to the monodromies (3.2) and the conformal-weight terms ($\Delta/(z-\omega)^2$) in the operator products, we define the modes of the stress tensors and twisted currents as follows:

$$\hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) = \sum_{m \in \mathbb{Z}} \hat{L}_{\hat{j}\hat{j}}(m + \frac{\hat{j}}{f_j(\sigma)}) z^{-(m + \frac{\hat{j}}{f_j(\sigma)}) - 2}, \quad (4.1a)$$

$$\hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) = \sum_{m \in \mathbb{Z}} \hat{J}_{n(r)\mu\hat{j}\hat{j}}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) z^{-(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) - 1}. \quad (4.1b)$$

This gives immediately the mode form of the orbifold Virasoro generators

$$\begin{aligned} \hat{L}_{\hat{j}\hat{j}}(m + \frac{\hat{j}}{f_j(\sigma)}) &= \sum_{n(r)\mu\nu} \frac{\mathcal{G}_{n(r)\mu; -n(r), \nu}(\sigma)}{2f_j(\sigma)} \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} \times \\ &\quad \times : \hat{J}_{n(r)\mu\hat{\ell}\hat{j}}(p + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_j(\sigma)}) \hat{J}_{-n(r), \nu, \hat{j}-\hat{\ell}, j}(m - p - \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}-\hat{\ell}}{f_j(\sigma)}) : \end{aligned} \quad (4.2)$$

and the mode periodicities

$$\hat{L}_{\hat{j}\pm f_j(\sigma),j}(m + \frac{\hat{j}\pm f_j(\sigma)}{f_j(\sigma)}) = \hat{L}_{\hat{j}j}(m \pm 1 + \frac{\hat{j}}{f_j(\sigma)}), \quad (4.3a)$$

$$\begin{aligned} \hat{J}_{n(r)\pm\rho(\sigma),\mu\hat{j}j}(m + \frac{n(r)\pm\rho(\sigma)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) &= \hat{J}_{n(r)\mu,\hat{j}\pm f_j(\sigma),j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}\pm f_j(\sigma)}{f_j(\sigma)}) \\ &= \hat{J}_{n(r)\mu\hat{j}j}(m \pm 1 + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}). \end{aligned} \quad (4.3b)$$

The operator-product normal-ordered forms (4.2) of the orbifold Virasoro generators are not as useful as the mode-normal ordered forms we shall obtain for these generators later.

We give next the twisted mode algebras of sector σ

$$\begin{aligned} &[\hat{L}_{\hat{j}j}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{L}_{\hat{\ell}\ell}(n + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ &= \delta_{j\ell} \left\{ (m - n + \frac{\hat{j}-\hat{\ell}}{f_j(\sigma)}) \hat{L}_{\hat{j}-\hat{\ell},j}(m + n + \frac{\hat{j}+\hat{\ell}}{f_j(\sigma)}) \right. \\ &\quad \left. + \frac{26f_j(\sigma)}{12} (m + \frac{\hat{j}}{f_j(\sigma)}) ((m + \frac{\hat{j}}{f_j(\sigma)})^2 - 1) \delta_{m+n+\frac{\hat{j}+\hat{\ell}}{f_j(\sigma)},0} \right\}, \end{aligned} \quad (4.4a)$$

$$\begin{aligned} &[\hat{L}_{\hat{j}j}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{J}_{n(r)\mu\hat{\ell}\ell}(n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ &= -\delta_{j\ell} (n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}) \hat{J}_{n(r)\mu,\hat{j}+\hat{\ell},j}(m + n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}+\hat{\ell}}{f_j(\sigma)}), \end{aligned} \quad (4.4b)$$

$$\begin{aligned} &[\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}), \hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(s)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ &= \delta_{j\ell} f_j(\sigma) (m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \delta_{n(r)+n(s),0 \bmod \rho(\sigma)} \\ &\quad \times \delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)}+\frac{\hat{j}+\hat{\ell}}{f_j(\sigma)},0} \mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma), \end{aligned} \quad (4.4c)$$

which are obtained by standard [11] orbifold contour methods from the twisted operator products (3.4) and the mode expansions (4.1) of the operators. The reader will recognize in particular the general orbifold Virasoro algebra (4.4a) obtained earlier¹ in Ref. [6] and quoted in Eq. (1.3) of the Introduction. We remind the reader of the ranges given in Eqs. (2.3),(2.4) for the quantum numbers $\hat{j}j$ ($H(\text{perm})_K$) and $n(r)\mu$ (H'_{26}) which appear in this result, as well as the definition of the twisted metric $\mathcal{G}_j(\sigma)$ in Eq. (2.8b).

In further detail, the orbifold Virasoro algebra (4.4a) of sector σ is semisimple with respect to the cycles j of each $W(\sigma) \in H(\text{perm})_K$, and each cycle has its own integral Virasoro subalgebra at cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$:

$$\begin{aligned} [\hat{L}_{0j}(m), \hat{L}_{0\ell}(n)] &= \delta_{j\ell} \left\{ (m - n) \hat{L}_{0j}(m + n) + \frac{26f_j(\sigma)}{12} m(m^2 - 1) \delta_{m+n,0} \right\}, \\ j, \ell &= 0, 1, \dots, N(\sigma) - 1, \end{aligned} \quad (4.5)$$

¹The general orbifold Virasoro algebra [4.4a] was first obtained in the WZW permutation orbifolds [15] with $26 \rightarrow c_g$, where c_g is the central charge of the affine-Sugawara construction [22] on Lie g .

where $N(\sigma)$ is the number of cycles in sector σ . The total Virasoro generators of sector σ are obtained by summing over the cycles of the sector

$$\hat{L}_\sigma(m) = \sum_j \hat{L}_{0j}(m), \quad \hat{c}(\sigma) = \sum_j \hat{c}_j(\sigma) = 26K \quad (4.6a)$$

$$[\hat{L}_\sigma(m), \hat{L}_\sigma(n)] = (m - n)\hat{L}_\sigma(m + n) + \frac{26K}{12}m(m^2 + 1)\delta_{m+n,0}, \quad (4.6b)$$

where we have used the cycle sum rule in Eq. (2.3c) to obtain the sector central charges $\hat{c}(\sigma)$.

Together, the twisted mode algebras (4.4) and the extended physical-state conditions (1.3a) form what we will call the *general cycle dynamics* of the matter in cycle j of sector σ . (The cycle dynamics includes the composite structure (4.2) of the orbifold Virasoro generators, but we remind that a more useful form of this structure will be obtained in the following section.)

We conclude with some comments on the applicability of the general cycle dynamics to the sectors of the three families (1.1) of orbifold-string theories of permutation-type. These results are complete as they stand for the twisted open string systems, including the open-string analogues (1.1b) of the generalized permutation orbifolds. The open-string sectors of the orientation orbifolds (1.1c) are included in the special case $K = f_0(\sigma) = 2$ and $a_2 = 17/8$ of these results at $\hat{c}(\sigma) = 52$. Right-mover copies of the cycle dynamics must be added to describe the generalized permutation orbifolds in Eq. (1.1a). The cycle dynamics of both open- and closed-string sectors at $\hat{c}(\sigma) = 52$ were described earlier in Ref. [3]. Finally, the cycle dynamics of the closed-string sectors of the orientation orbifolds (the ordinary space-time orbifolds $U(1)^{26}/H'_{26}$) are obtained with a right-mover copy by choosing $K = f_0(\sigma) = 1$ and therefore the conventional intercept $a_1 = 1$ at $\hat{c}(\sigma) = 26$.

5 Mode normal-ordering

To obtain a more useful form of the orbifold Virasoro generators $\{L_{\hat{j}j}\}$, the next step in the orbifold program is the introduction of *mode normal-ordering*

$$\begin{aligned} & :\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)})\hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(s)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}):_M \\ & \equiv \theta((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \geq 0)\hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(s)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)})\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \\ & + \theta((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) < 0)\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)})\hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(s)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}) \quad (5.1) \end{aligned}$$

to replace the operator-product normal-ordering in Eq. (4.2).

The reordering is somewhat intricate, so we will sketch the intermediate steps. From the definition (5.1) of mode normal-ordering and the commutator (4.4c) of two twisted currents, we obtain first the relation between the product of two modes and the mode normal-ordered

product of the modes as follows:

$$\begin{aligned}
& \hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}) \\
&= : \hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}) :_M \\
&+ \theta((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) \delta_{j\ell} f_j(\sigma) (m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \\
&\quad \cdot \delta_{n(r)+n(s), 0 \bmod \rho(\sigma)} \delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)}+\frac{\hat{j}+\hat{\ell}}{f_j(\sigma)}, 0} \mathcal{G}_{n(r)\mu; -n(r), \nu}(\sigma). \quad (5.2)
\end{aligned}$$

Then using the mode expansions and the $\hat{J}\hat{J}$ operator product in Eq. (3.4c), one straightforwardly obtains the following exact relation between the two types of normal-ordering of two local currents:

$$\begin{aligned}
& : \hat{J}_{n(r)\mu\hat{j}j}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\ell}(\omega, \sigma) : - : \hat{J}_{n(r)\mu\hat{j}j}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\ell}(\omega, \sigma) :_M \\
&= f_j(\sigma) \delta_{j\ell} \mathcal{G}_{n(r)\mu; -n(r), \nu}(\sigma) \delta_{n(r)+n(s), 0 \bmod \rho(\sigma)} \delta_{\hat{j}+\hat{\ell}, 0 \bmod f_j(\sigma)} \\
&\quad \cdot \left[\frac{1}{z\omega} \left(\frac{\omega}{z} \right)^{X_{\hat{j}}} \left\{ X_{\hat{j}} \frac{z}{z-\omega} (\theta(0 \leq X_{\hat{j}} < 1) + \frac{z}{\omega} \theta(1 \leq X_{\hat{j}} < 2)) \right. \right. \\
&\quad \left. \left. + \frac{z\omega}{(z-\omega)^2} \theta(0 \leq X_{\hat{j}} < 1) - \frac{z(z-2\omega)}{\omega^2} \theta(1 \leq X_{\hat{j}} < 2) \right\} \right. \\
&\quad \left. - \frac{1}{(z-\omega)^3} \right] \quad (5.3a)
\end{aligned}$$

$$X_{\hat{j}} \equiv \frac{\bar{n}(r)}{\rho(\sigma)} + \frac{\bar{\hat{j}}}{f_j(\sigma)}, \quad 0 \leq X_{\hat{j}} < 2, \quad \forall \bar{n}(r), \bar{\hat{j}}. \quad (5.3b)$$

Here θ is the Heaviside function, and we have introduced the notational simplification $X_{\hat{j}} \equiv X_{\hat{j}n(r)}$. As $z \rightarrow \omega$, Eq. (5.3) gives the exact relation between the two types of local normal-ordered current bilinears

$$\begin{aligned}
& : \hat{J}_{n(r)\mu\hat{j}j}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\ell}(z, \sigma) : = : \hat{J}_{n(r)\mu\hat{j}j}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\ell}(z, \sigma) :_M \\
&+ \delta_{j\ell} f_j(\sigma) \delta_{\hat{j}+\hat{\ell}, 0 \bmod f_j(\sigma)} \mathcal{G}_{n(r)\mu; -n(r), \nu}(\sigma) \delta_{n(r)+n(s), 0 \bmod \rho(\sigma)} \\
&\quad \cdot \frac{1}{z^2} |1 - X_{\hat{j}}| (1 - |1 - X_{\hat{j}}|) \quad (5.4)
\end{aligned}$$

which is now in the desired form for application to the orbifold Virasoro generators.

With the relation (5.4), we can convert the operator-product normal-ordered forms (3.1) or (4.2) of the extended stress tensors and orbifold Virasoro generators to the following

mode-ordered forms

$$\hat{\theta}_{jj}(z, \sigma) = \frac{1}{2f_j(\sigma)} \sum_{n(r)\mu\nu} \mathcal{G}^{n(r)\mu; -n(r),\nu}(\sigma) \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} : \hat{J}_{n(r)\mu\hat{\ell}j}(z, \sigma) \hat{J}_{-n(r),\nu,\hat{j}-\hat{\ell},j}(z, \sigma) :_M + \frac{\delta_{j,0 \bmod f_j(\sigma)}}{z^2} \hat{\Delta}_{0j}(\sigma) \quad (5.5a)$$

$$\begin{aligned} \hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}) &= \frac{1}{2f_j(\sigma)} \sum_{n(r)\mu\nu} \mathcal{G}^{n(r)\mu; -n(r),\nu}(\sigma) \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} \times \\ &\times : \hat{J}_{n(r)\mu\hat{\ell}j}(p + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_j(\sigma)}) \hat{J}_{-n(r)\nu,\hat{j}-\hat{\ell},j}(m - p - \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}-\hat{\ell}}{f_j(\sigma)}) :_M \\ &+ \delta_{m+\frac{\hat{j}}{f_j(\sigma)},0} \hat{\Delta}_{0j}(\sigma) \end{aligned} \quad (5.5b)$$

where the shifts $\{\hat{\Delta}_{0j}(\sigma)\}$, given below, will be called the *conformal weights of cycle j in sector σ* . (These conformal weights were called the partial conformal weights in the early examples of Ref. [15].)

We give the explicit forms of the conformal weights $\{\hat{\Delta}_{0j}(\sigma)\}$ in a number of steps, which show various properties of these quantities:

$$\begin{aligned} \hat{\Delta}_{0j}(\sigma) &= \frac{1}{4} \sum_{n(r)\mu n(s)\nu} \mathcal{G}^{n(r)\mu; n(s)\nu}(\sigma) \mathcal{G}_{n(r)\mu; n(s)\nu}(\sigma) \sum_{\hat{j}=0}^{f_j(\sigma)-1} \times \\ &\times \{ \theta(0 \leq X_{\hat{j}} < 1) X_{\hat{j}}(1 - X_{\hat{j}}) + \theta(1 \leq X_{\hat{j}} < 2) (X_{\hat{j}} - 1)(2 - X_{\hat{j}}) \} \end{aligned} \quad (5.6a)$$

$$= \frac{1}{4} \sum_{n(r)\mu} \sum_{\hat{j}=0}^{f_j(\sigma)-1} \{ \theta(0 \leq X_{\hat{j}} < 1) X_{\hat{j}}(1 - X_{\hat{j}}) + \theta(1 \leq X_{\hat{j}} < 2) (X_{\hat{j}} - 1)(2 - X_{\hat{j}}) \} \quad (5.6b)$$

$$= \frac{1}{2} \sum_r \dim[n(r)] \sum_{\hat{j}=0}^{f_j(\sigma)-1} (1 - \frac{\bar{n}(r)}{\rho(\sigma)} - \frac{\bar{j}}{f_j(\sigma)}) \{ \frac{1}{2} (\frac{\bar{n}}{\rho(\sigma)} + \frac{\bar{j}}{f_j(\sigma)}) - \theta((\frac{\bar{n}}{\rho(\sigma)} + \frac{\bar{j}}{f_j(\sigma)}) \geq 1) \}. \quad (5.6c)$$

To obtain the second form (5.6b), we have used Eq. (2.8c) to do the sum on $n(s)\nu$, and the final step (5.6c) uses the degeneracy sum $\sum_{\mu} = \dim[\bar{n}(r)]$ in Eq. (2.4b). The bars on the quantities \hat{j} can be ignored here because the fundamental range is explicit in the summations. The second form in particular shows that the conformal weight of cycle j is nonnegative

$$\hat{\Delta}_{0j}(\sigma) \geq 0 \quad (5.7)$$

and we shall find stronger lower bounds below.

Other useful forms of the conformal weights of cycle j include:

$$\hat{\Delta}_{0j}(\sigma) = \frac{13}{12} \left(f_j(\sigma) - \frac{1}{f_j(\sigma)} \right) + \frac{1}{f_j(\sigma)} \hat{\delta}_{0j}(\sigma), \quad (5.8a)$$

$$\hat{\delta}_{0j}(\sigma) \equiv \frac{f_j(\sigma)}{2} \sum_r \dim[\bar{n}(r)] \hat{A}[\frac{\bar{n}(r)}{\rho(\sigma)}], \quad (5.8b)$$

$$\sum_r \dim[\bar{n}(r)] = 26, \quad (5.8c)$$

$$\begin{aligned} \hat{A}[\frac{\bar{n}(r)}{\rho(\sigma)}] \equiv & \left(\frac{\bar{n}(r)}{\rho(\sigma)} - \frac{1}{f_j(\sigma)} \right) \left(\theta\left(\frac{\bar{n}(r)}{\rho(\sigma)} \geq \frac{1}{f_j(\sigma)}\right) - \frac{f_j(\sigma)}{2} \frac{\bar{n}(r)}{\rho(\sigma)} \right) \\ & + \sum_{\hat{j}=2}^{f_j(\sigma)-1} \left(\frac{\bar{n}(r)}{\rho(\sigma)} - \frac{\hat{j}}{f_j(\sigma)} \right) \theta\left(\frac{\bar{n}(r)}{\rho(\sigma)} \geq \frac{\hat{j}}{f_j(\sigma)}\right), \end{aligned} \quad (5.8d)$$

$$= \frac{f_j(\sigma)}{2} \sum_{\hat{j}=0}^{f_j(\sigma)-1} \left(\frac{\bar{n}(r)}{\rho(\sigma)} - \frac{\hat{j}}{f_j(\sigma)} \right) \left(\frac{\hat{j}+1}{f_j(\sigma)} - \frac{\bar{n}(r)}{\rho(\sigma)} \right) \theta\left(\frac{\hat{j}}{f_j(\sigma)} \leq \frac{\bar{n}(r)}{\rho(\sigma)} < \frac{\hat{j}+1}{f_j(\sigma)}\right). \quad (5.8e)$$

In what follows, we will refer to the quantity $\hat{\delta}_{0j}(\sigma)$ as the *conformal-weight shift* of cycle j in sector σ . The first form of the function \hat{A} in Eq. (5.8d) follows directly from Eq. (5.6c), and is easier to evaluate explicitly for small cycle length $f_j(\sigma)$. The second form of \hat{A} in Eq. (5.8e) follows by induction from the first form, and shows that

$$\hat{A}[\frac{\bar{n}(r)}{\rho(\sigma)}] \geq 0 \quad \implies \quad \hat{\delta}_{0j}(\sigma) \geq 0 \quad \implies \quad \hat{\Delta}_{0j}(\sigma) \geq \frac{13}{12} \left(f_j(\sigma) - \frac{1}{f_j(\sigma)} \right). \quad (5.9)$$

Moreover \hat{A} is a continuous function, periodic in the H'_{26} -fraction (\bar{n}/ρ) with period $(1/f_j(\sigma))$. The only zeroes of \hat{A} are at the values

$$f_j(\sigma) \frac{\bar{n}(r)}{\rho(\sigma)} \in \mathbb{Z}_{\geq 0} \quad (5.10)$$

and the maximum value in each cell is $(1/8f_j(\sigma))$.

Let us check our general result (5.8) for the previously-studied cases [3] of $H(\text{perm})_2 = \mathbb{Z}_2$ or $\mathbb{Z}_s(\text{w.s.})$, i.e. for the generalized \mathbb{Z}_2 -permutation orbifolds or the twisted open-string sectors of the orientation orbifolds. Choosing for either case the single nontrivial element with a single cycle $j = 0$ of length $f_0(\sigma) = 2$, we can easily² do the sum explicitly over $\bar{j} = 0, 1$ to obtain

$$\hat{\Delta}_{00}(\sigma) = \frac{13}{8} + \sum_r \dim[\bar{n}(r)] \left(\frac{\bar{n}(r)}{\rho(\sigma)} - \frac{1}{2} \right) \left(\theta\left(\frac{\bar{n}(r)}{\rho(\sigma)} \geq \frac{1}{2}\right) - \frac{\bar{n}(r)}{\rho(\sigma)} \right) \geq \frac{13}{8}. \quad (5.11)$$

This is in agreement with the result in Eq. (2.3d) of Ref. [3].

²On the other hand, we find a repeated typo in Eqs. (3.36c) and (3.38b) of the earlier Ref. [18]: The terms $(n(r)/2\rho(\sigma))^2$ in each of these equations should read simply $(n(r)/\rho(\sigma))^2$, without the 2 in the denominator.

There is one more relation between the quantities discussed here which will be useful to record

$$\hat{\Delta}_{0j}(\sigma) - \hat{a}_{f_j(\sigma)} = \frac{1}{f_j(\sigma)}(\hat{\delta}_{0j}(\sigma) - 1) \quad (5.12)$$

where $\hat{a}_{f_j(\sigma)}$ is the *intercept* of cycle j in sector σ in the extended physical-state condition (1.3a).

We close this section with a simple application of our results here to the *twist-field state* $|0\rangle_{j\sigma}$ of cycle j in sector σ

$$\hat{J}_{n(r)\mu\hat{j}j}((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) |0\rangle_{j\sigma} = 0, \quad (5.13a)$$

$$\{\hat{L}_{\hat{j}j}((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) - \hat{\Delta}_{0j}(\sigma)\delta_{m+\frac{\hat{j}}{f_j(\sigma)},0}\} |0\rangle_{j\sigma} = 0, \quad (5.13b)$$

$$(\hat{L}_\sigma(m \geq 0) - \hat{\Delta}_\sigma\delta_{m,0}) |0\rangle_\sigma = 0, \quad (5.13c)$$

$$|0\rangle_\sigma \equiv \bigotimes_j |0\rangle_{j\sigma}, \quad \hat{\Delta}_\sigma \equiv \sum_j \hat{\Delta}_{0j}(\sigma). \quad (5.13d)$$

The first line in Eq. (5.13) defines this state, while the succeeding lines then follow from the mode-normal-ordered form (5.5b) of the orbifold Virasoro generators. Although the twist-field state is closely related to the physical ground-state of each cycle (see Sec. 7), we should emphasize that the twist-field state itself is not generally a physical state.

6 First discussion of the zero modes

To analyze the spectral problems associated to the extended physical-state conditions (1.3a), we need to separate out the *zero modes* $\{\hat{J}_j(0)_\sigma\}$ of cycle j in sector σ from the doubly-twisted currents

$$\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}).$$

We remind that $\mu = \mu(n(r))$ is the degeneracy index of the spectral index $n(r)$ of each element of the 26-dimensional automorphism group H'_{26} , while $\{\hat{j}j\}$ record the cycle-basis of each element of $H(\text{perm})_K$. The zero modes are special cases of what we will call the *integer-moded sequences* $\{\hat{J}_j(m)_\sigma\}$ of cycle j in sector σ .

From the twisted current algebra (4.4c), we know that the zero modes commute with all the currents, including themselves

$$[\{\hat{J}_j(0)_\sigma\}, \hat{J}_{n(r)\mu\hat{\ell}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)})] = 0, \quad \forall j, \ell, \sigma \quad (6.1)$$

so we may alternatively refer to the zero modes as the *momenta* of cycle j in sector σ . Then it is natural to define the number of zero modes in each cycle as the *target space-time dimension* of cycle j in sector σ :

$$\hat{D}_j(\sigma) \equiv \dim\{\hat{J}_j(0)_\sigma\}. \quad (6.2)$$

The target space-time interpretation is a topic of central importance in the new string theories. In this paper, we confine ourselves only to general properties of the space-time dimensions, across all the bosonic prototypes (1.1) of the orbifold-string theories of permutation-type. The general formulae developed here are however quite model-dependent, involving the choice of subgroup H'_{26} and the particular element $\omega(\sigma) \in H'_{26}$. Beyond the simple examples of trivial H'_{26} in Sec. 10, we will return to study the target space-times of large classes of specific models in succeeding papers of this series.

From the total mode number of the doubly-twisted currents, it is straightforward to determine that the following conditions are necessary and sufficient for integer-moded sequences and hence zero modes in cycle j of sector σ :

$$\{\hat{J}_j(0)_\sigma\} : f_j(\sigma) \frac{\bar{n}(r)}{\rho(\sigma)} \in \mathbb{Z}_{\geq 0}, \quad (6.3a)$$

$$\bar{n}(r) = \frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)} \quad \text{such that } \hat{j}' = 0, 1, \dots, f_j(\sigma) - 1, \bar{n}(r) \in \{0, 1, \dots, \rho(\sigma) - 1\}. \quad (6.3b)$$

The conditions in (6.3b) are a more detailed statement of the condition in (6.3a). The explicit solutions of these conditions depend on the cycle length $f_j(\sigma)$ in sector σ of $H(\text{perm})_K$, as well as the details of $\omega(\sigma) \in H'_{26}$ reflected in the ratio $\frac{\bar{n}(r)}{\rho(\sigma)}$. We emphasize that the condition (6.3a) is the same condition under which the function $\hat{A}[\frac{\bar{n}(r)}{\rho(\sigma)}] = 0$ (see Eq. (5.10)), so that the integer-moded sequences do not contribute to the conformal-weight shift $\hat{\delta}_{0j}(\sigma)$ of cycle j in sector σ . This phenomenon is familiar in ordinary untwisted string theory, where all sequences are integer-moded and there are no conformal-weight shifts.

Using the conditions (6.3), we can give a qualitative sketch of the integer-moded sequences and zero modes as follows. We begin by noting that there are exactly two possible types of such sequences, the first of which

$$\text{type I: } \hat{J}_{0\mu(0)0j}(m) \xrightarrow{\bar{n}=0} \hat{J}_{0\mu(0)0j}(0) \quad (6.4)$$

is found if and only if $\bar{n} = 0$ occurs in the spectrum of $\omega(\sigma) \in H'_{26}$. This type occurs for example (see Sec. 10) in each cycle of every sector of the “pure” permutation orbifolds with trivial H'_{26} . The second, distinct type that can arise is the following family:

$$\text{type II: } \hat{J}_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu(\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}), f_j(\sigma)-\hat{j}', j}(m+1) \xrightarrow{m=-1} \hat{J}_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu(\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}), f_j(\sigma)-\hat{j}', j}(0), \quad (6.5a)$$

$$\hat{j}' = 1, \dots, f_j(\sigma) - 1. \quad (6.5b)$$

Because the spectral index $\bar{n}(r)$ is an integer, this family occurs only when $\rho(\sigma)$ is a multiple of $f_j(\sigma)$ or vice-versa. Examples of this type with $(f_j(\sigma) = 2, \rho(\sigma) = \text{even})$ have been discussed in Ref. [3], and we will discuss both of these types more systematically in succeeding papers.

For our discussion below, it will be convenient to introduce a formal Heaviside function for the momenta:

$$\theta\{\hat{J}_j(0)_\sigma\} \equiv \begin{cases} 1 & \text{when } \omega(\sigma) \in H'_{26} \text{ allows the zero mode in } j\sigma, \\ 0 & \text{otherwise.} \end{cases} \quad (6.6)$$

This allows us to write formal expressions for the number of target space-time dimensions and the “momentum-squared” operator of cycle j in sector σ :

$$\hat{D}_j(\sigma) = \dim\{\hat{J}_j(0)_\sigma = \sum_{\bar{n}(r)\mu\hat{j}'} \theta\{\hat{J}_j(0)_\sigma\} \quad (6.7a)$$

$$\begin{aligned} \hat{P}_j^2(\sigma) \equiv & - \sum_{\mu,\nu} \theta\{\hat{J}_j(0)_\sigma\} \{ \mathcal{G}^{0\mu;0\nu}(\sigma) \hat{J}_{0\mu 0j}(0) \hat{J}_{0\nu 0j}(0) \\ & + \sum_{\hat{j}'=1}^{f_j(\sigma)-1} \mathcal{G}_{\frac{\rho_{\hat{j}'}}{f_j}, \mu; -\frac{\rho_{\hat{j}'}}{f_j}, \nu}(\sigma) \hat{J}_{\frac{\rho_{\hat{j}'}}{f_j}, \mu, f_j - \hat{j}', j}(0) \hat{J}_{-\frac{\rho_{\hat{j}'}}{f_j}, \nu, \hat{j}' - f_j, j}(0) \}. \end{aligned} \quad (6.7b)$$

For brevity, we have omitted here the sector label σ in both $f_j(\sigma)$ and $\rho(\sigma)$. The momentum-squared operator in Eq. (6.7b) will play a central role in the following analysis of the extended physical-state conditions.

7 The extended physical-state conditions

We begin this section by recalling the extended physical-state conditions for the open-string analogues of the generalized permutation orbifolds

$$\left[\frac{\mathrm{U}(1)^{26K}}{H_+} \right]_{\text{open}}, \quad H_+ \subset H(\text{perm})_K \times H'_{26}, \quad (7.1)$$

which include the twisted open-string sectors of the orientation orbifolds when $K = 2$. As obtained in the BRST quantization of Ref. [6] and quoted above in Eq. (1.3), these conditions read

$$(\hat{L}_{\hat{j}j}((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) - \hat{a}_{f_j(\sigma)} \delta_{m + \frac{\hat{j}}{f_j(\sigma)}, 0}) |\chi(\sigma)\rangle_j = 0, \quad (7.2a)$$

$$\hat{c}_j(\sigma) = 26f_j(\sigma), \quad \hat{a}_{f_j(\sigma)} = \frac{13f_j^2(\sigma) - 1}{12f_j(\sigma)} \quad (7.2b)$$

$$\bar{j} = 0, 1, \dots, f_j(\sigma) - 1, \quad j = 0, 1, \dots, N(\sigma) - 1 \quad (7.2c)$$

where $\hat{c}_j(\sigma)$ and $\hat{a}_{f_j(\sigma)}$ are respectively the cycle central charge and the intercept of cycle j in sector σ . The integer $N(\sigma)$ in Eq. (7.2c) is the number of cycles in sector σ , while the orbifold Virasoro generators $\{\hat{L}_{\hat{j}j}\}$ should now be taken in the mode normal-ordered form (5.5b).

For each cycle j in every sector σ , this system can be decomposed into the extended gauge conditions

$$\hat{L}_{\hat{j}j}((m + \frac{\hat{j}}{f_j(\sigma)}) > 0) |\chi(\sigma)\rangle_j = 0 \quad (7.3)$$

and the spectral subproblem for cycle j of sector σ :

$$(\hat{L}_{0j}(0) - \hat{a}_{f_j(\sigma)}) |\chi(\sigma)\rangle_j = 0, \quad (7.4a)$$

$$\hat{L}_{0j}(0) = \frac{1}{2f_j(\sigma)}(-\hat{P}_j^2(\sigma) + \hat{R}_j(\sigma)) + \hat{\Delta}_{0j}(\sigma), \quad (7.4b)$$

$$\begin{aligned} \hat{R}_j(\sigma) \equiv \sum'_{n(r)\mu\nu} \mathcal{G}^{n(r)\mu; -n(r)\nu}(\sigma) \sum_{\hat{j}=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} \times \\ \times : \hat{J}_{n(r)\mu\hat{j}j}(p + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \hat{J}_{-n(r),\nu,-\hat{j},j}(-p - \frac{n(r)}{\rho(\sigma)} - \frac{\hat{j}}{f_j(\sigma)}) :_M. \end{aligned} \quad (7.4c)$$

The momentum-squared operator (6.7b) appears now among the terms of Eq. (7.4b), and the primed sum in the generalized number operator $\hat{R}_j(\sigma)$ denotes omission of the zero modes.

So long as the cycle-momenta $\{\hat{J}_j(0)_\sigma\}$ are not an empty set, the *physical ground-state* of cycle j in sector σ is the $\{\hat{J}_j(0)_\sigma\}$ -boosted twist-field state $|0, \hat{J}_j(0)\rangle_\sigma$

$$\hat{J}_{n(r)\mu\hat{j}j}((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) > 0) |0, \hat{J}_j(0)\rangle_\sigma = 0, \quad (7.5a)$$

$$\hat{R}_j(\sigma) |0, \hat{J}_j(0)\rangle_\sigma = \hat{L}_{jj}((m + \frac{\hat{j}}{f_j(\sigma)}) > 0) |0, \hat{J}_j(0)\rangle_\sigma = 0, \quad (7.5b)$$

$$\hat{P}_j^2(\sigma) |0, \hat{J}_j(0)\rangle_\sigma = \hat{P}_j^2(\sigma)_{(0)} |0, \hat{J}_j(0)\rangle_\sigma, \quad (7.5c)$$

$$\hat{P}_j^2(\sigma)_{(0)} = 2(\hat{\delta}_{0j}(\sigma) - 1) \geq -2 \quad (7.5d)$$

where the *ground-state momentum-squared* of cycle j in sector σ is given in Eq. (7.5d). To obtain this result, we used Eqs. (7.4a,b), and the relation (5.12) between the fundamental constants of the cycle. The explicit form of the conformal-weight shift $\hat{\delta}_{0j}(\sigma) \geq 0$ is given in Eq. (5.8b), and we see from Eq. (7.5d) that the conformal-weight shift does indeed measure a shift from the ground-state momentum-squared $P^2 = -2$ of an ordinary untwisted open string.

According to Eqs. (4.4b) and (7.4b), the excited states of cycle j in sector σ exhibit the *level-spacing*

$$\Delta(\hat{P}_j^2(\sigma)) = \Delta(\hat{R}_j(\sigma)) = 2f_j(\sigma) |m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}|. \quad (7.6)$$

These are the increments of mass-squared associated to the addition of any negatively-moded current

$$\hat{J}_{n(r)\mu\hat{j}j}((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) < 0)$$

to products of other such currents on the ground-state of cycle j in sector σ .

We turn next to the extended physical-state conditions of all the closed-string sectors of the generalized permutation orbifolds

$$\frac{\text{U}(1)^{26K}}{H_+}, \quad H_+ \subset H(\text{perm})_K \times H'_{26} \quad (7.7)$$

whose cycles also live at cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$ and sector central charge $\hat{c}(\sigma) = 26K$. In these cases we have a left- and right-mover copy of the extended physical-state conditions (7.2), which we write as

$$\begin{aligned}\hat{L}_{jj}^L((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) |\chi(\sigma)\rangle_j &= \hat{L}_{jj}^R((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) |\chi(\sigma)\rangle_j \\ &= \hat{a}_{f_j(\sigma)} \delta_{m + \frac{\hat{j}}{f_j(\sigma)}, 0} |\chi(\sigma)\rangle_j,\end{aligned}\tag{7.8a}$$

$$\hat{a}_{f_j(\sigma)} = \frac{13f_j^2(\sigma) - 1}{12f_j(\sigma)},\tag{7.8b}$$

$$\hat{L}_{0j}^L(0) = \frac{1}{2f_j(\sigma)} (\hat{P}_j^2(\sigma)^L + \hat{R}_j(\sigma)^L + \hat{\Delta}_{0j}(\sigma)),\tag{7.8c}$$

$$\hat{L}_{0j}^R(0) = \frac{1}{2f_j(\sigma)} (\hat{P}_j^2(\sigma)^R + \hat{R}_j(\sigma)^R + \hat{\Delta}_{0j}(\sigma)).\tag{7.8d}$$

The extended Virasoro generators $\{\hat{L}^L\}$ and $\{\hat{L}^R\}$ involve the twisted left- and right-mover currents $\{\hat{J}^L\}$ and $\{\hat{J}^R\}$ respectively.

Following Ref. [3], we study only the case of decompactified zero modes, with the ordinary left-right identifications:

$$\hat{J}_j^R(0)_\sigma = \hat{J}_j^L(0)_\sigma = \frac{1}{\sqrt{2}} \hat{J}_j(0)_\sigma,\tag{7.9a}$$

$$\hat{P}_j^2(\sigma)^R = \hat{P}_j^2(\sigma)^L = \frac{1}{2} \hat{P}_j^2(\sigma).\tag{7.9b}$$

The closed-string momenta $\{\hat{J}_j(0)_\sigma\}$ and momentum-squared $\hat{P}_j^2(\sigma)$ appear on the right side of Eqs. (7.9 a,b), and $\hat{P}_j^2(\sigma)$ has exactly the form (6.7b) when expressed in terms of $\{\hat{J}_j(0)_\sigma\}$. Then the extended physical-state conditions (7.8) can be put in the form

$$\hat{P}_j^2(\sigma) |\chi(\sigma)\rangle_j = 2(\hat{P}_j^2(\sigma)_{(0)} + \hat{R}_j^L(\sigma)) |\chi(\sigma)\rangle_j,\tag{7.10a}$$

$$(\hat{R}_j^R(\sigma) - \hat{R}_j^L(\sigma)) |\chi(\sigma)\rangle_j = 0,\tag{7.10b}$$

$$\hat{L}_{jj}^R((m + \frac{\hat{j}}{f_j(\sigma)}) > 0) |\chi(\sigma)\rangle_j = \hat{L}_{jj}^L((m + \frac{\hat{j}}{f_j(\sigma)}) > 0) |\chi(\sigma)\rangle_j = 0\tag{7.10c}$$

where $\hat{P}_j^2(\sigma)_{(0)}$ is defined in Eq. (7.5d), and Eq. (7.10b) is the level-matching condition for cycle j in sector σ . Assuming again that the momenta are not an empty set, we find that the physical closed-string ground-state $|0, \hat{J}_j(0)\rangle_\sigma$ of cycle j has ground-state momentum-squared

$$\hat{P}_j^2(\sigma)^{\text{closed}}(0) = 2\hat{P}_j^2(\sigma)_{(0)} = 4(\hat{\delta}_{0j}(\sigma) - 1) \geq -4\tag{7.11}$$

that is, twice the ground-state momentum-squared of the corresponding twisted open string.

In what follows we will explicitly discuss only the twisted open-string cases, but the corresponding closed-string cases can easily be obtained from the results above.

8 The reduced formulation at $c_j(\sigma) = 26$

Recall that the $\hat{c}(\sigma) = 52$ physical spectral problems have an equivalent, reduced description [3,4] at reduced central charge $c(\sigma) = 26$. In this and the following section, we generalize this result to include all the bosonic prototypes (1.1) at sector central charge $\hat{c}(\sigma) = 26K$ and cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$. In particular, we find the equivalent, reduced formulation of each cycle j at *reduced cycle central charge* $c_j(\sigma) = 26$, independent of the cycle. We emphasize with the earlier references that *this equivalence holds only for the cycle dynamics of the orbifold-string theories as described by the extended physical-state conditions*, and *not* for the underlying orbifold CFT's themselves. One advantage of the original description at $\hat{c}(\sigma) = 26K$ is locality [1], which provides twisted local vertex operators [13,15,18,21,4,5]. With Ref. [3] we shall see however that the zero modes and target space-time dimensions are *invariant* under the reduction, and we shall emphasize in succeeding papers that the target space-time properties of the theories are more easily studied in the reduced description.

In this paper we organize the discussion of the equivalent, reduced formulation into two parts. In the present section we work out the operators of the reduced formulation at $c_j(\sigma) = 26$, leaving for Sec. 9 the reduced physical-state conditions and the equivalence of the two formulations at the string level.

The reduced (unhatted) operators of cycle j in sector σ are defined by the following map

$$L_j(M_j) \equiv f_j(\sigma) \hat{L}_{\hat{j}j}(m + \frac{\hat{j}}{f_j(\sigma)}) - \frac{13}{12}(f_j(\sigma)^2 - 1) \delta_{m + \frac{\hat{j}}{f_j(\sigma)}, 0}, \quad (8.1a)$$

$$J_{n(r)\mu j}(M_j + f_j(\sigma) \frac{n(r)}{\rho(\sigma)}) \equiv \hat{J}_{n(r)\mu \hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}), \quad (8.1b)$$

$$M_j \equiv f_j(\sigma)m + \bar{\hat{j}} \in \mathbb{Z} \quad (8.1c)$$

in terms of the hatted operators above. Since $m \in \mathbb{Z}$ and $\bar{\hat{j}} \in 0, 1, \dots, f_j(\sigma) - 1$, the capitalized quantities M_j cover the integers once for each cycle j , and indeed the map is one-to-one at each fixed (j, σ) . This map is in fact a modest generalization of the (inverse of) the order- λ orbifold-induction procedure of Borisov, Halpern, and Schweigert [7].

Then we find from Eq. (4.4) the explicit algebra of the reduced operators:³

$$[L_j(M), L_\ell(N)] = \delta_{j\ell} \{ (M - N)L_j(M + N) + \frac{26}{12}M(M^2 - 1)\delta_{M+N,0} \}, \quad (8.2a)$$

$$[L_j(M), J_{n(r)\mu\ell}(N + f_\ell(\sigma)\frac{n(r)}{\rho(\sigma)})] = -\delta_{j\ell}(N + f_\ell(\sigma)\frac{n(r)}{\rho(\sigma)})J_{n(r)\mu\ell}(M + N + f_\ell(\sigma)\frac{n(r)}{\rho(\sigma)}), \quad (8.2b)$$

$$\begin{aligned} & [J_{n(r)\mu j}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}), J_{n(s)\nu\ell}(N + f_\ell(\sigma)\frac{n(s)}{\rho(\sigma)})] \\ &= \delta_{j\ell}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)})\delta_{n(r)+n(s), 0 \bmod \rho(\sigma)}\delta_{M+N+f_j(\sigma)\frac{n(r)+n(s)}{\rho(\sigma)}, 0}\mathcal{G}_{n(r)\mu; -n(r)\nu}(\sigma), \end{aligned} \quad (8.2c)$$

$$J_{n(r)\pm\rho(\sigma), \mu j}(m + f_j(\sigma)\frac{n(r)\pm\rho(\sigma)}{\rho(\sigma)}) = J_{n(r)\mu}(m \pm f_j(\sigma) + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}). \quad (8.2d)$$

In particular, Eq. (8.2a) shows that the reduced generators $\{L_j(M)\}$ satisfy an ordinary Virasoro algebra with reduced cycle central charge $c_j(\sigma) = 26$ for each cycle j in any sector σ . The total reduced Virasoro generators of sector σ

$$L_\sigma(M) \equiv \sum_j L_j(M) \quad (8.3)$$

are then also Virasoro with reduced sector central charge

$$c(\sigma) = \sum_j c_j(\sigma) = 26N(\sigma) \quad (8.4)$$

where $N(\sigma)$ is the number of cycles in sector σ .

With Eq. (5.5b), the map also gives the explicit form of the reduced Virasoro generators of each cycle at $c_j(\sigma) = 26$ in terms of the reduced currents:

$$\begin{aligned} L_j(M) &= \delta_{M,0}\hat{\delta}_{0j}(\sigma) + \frac{1}{2} \sum_{n(r)\mu\nu} \mathcal{G}^{n(r)\mu; -n(r)\nu}(\sigma) \sum_{p \in \mathbb{Z}} \times \\ &\quad \times :J_{n(r)\mu j}(P + f_j(\sigma)\frac{n(r)}{\rho(\sigma)})J_{-n(r), \nu j}(M - P - f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) :_M, \end{aligned} \quad (8.5a)$$

$$\begin{aligned} \hat{\delta}_{0j}(\sigma) &= \frac{1}{2} \sum_r \dim[\bar{n}(r)] \left\{ (f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} - 1)(\theta(f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} \geq 1) - f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)}) \right. \\ &\quad \left. + \sum_{\hat{j}=2}^{f_j(\sigma)-1} (f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} - \hat{j})\theta(f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} \geq \hat{j}) \right\}, \end{aligned} \quad (8.5b)$$

$$= \frac{1}{4} \sum_r \dim[\bar{n}(r)] \sum_{\hat{j}=0}^{f_j(\sigma)-1} (f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} - \hat{j})(\hat{j} + 1 - \frac{f_j(\sigma)\bar{n}(r)}{\rho(\sigma)})\theta(\hat{j} \leq \frac{f_j(\sigma)\bar{n}(r)}{\rho(\sigma)} < \hat{j} + 1), \quad (8.5c)$$

$$j = 0, 1, \dots, N(\sigma) - 1, \quad \sum_j f_j(\sigma) = K, \quad \sum_r \dim[\bar{n}(r)] = 26. \quad (8.5d)$$

³ In Eq. (4.2e) of Ref. [3], there is a missing factor $(M + 2\frac{n(r)}{\rho(\sigma)})$, which is now included in (8.2c) when $f_j(\sigma) = 2$.

Here the expressions for the conformal-weight shifts $\hat{\delta}_{0j}(\sigma)$ are the same as those given in Eq. (5.8), now slightly rearranged to emphasize the scaling into the characteristic ratio $(f_j(\sigma)\frac{n(r)}{\rho(\sigma)})$ of the reduced formulation. The explicit form of the mode-normal ordering here

$$\begin{aligned} & :J_{n(r)\mu j}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)})J_{n(s)\nu\ell}(N + f_\ell(\sigma)\frac{n(s)}{\rho(\sigma)}) :_M \\ & = \theta((M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) \geq 0)J_{n(s)\nu\ell}(N + f_\ell(\sigma)\frac{n(s)}{\rho(\sigma)})J_{n(r)\mu j}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) \\ & \quad + \theta((M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) < 0)J_{n(r)\mu j}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)})J_{n(s)\nu\ell}(N + f_\ell(\sigma)\frac{n(s)}{\rho(\sigma)}) \quad (8.6) \end{aligned}$$

is also obtained as the image of the original mode-ordering in Eq. (5.2). This result reflects the simple fact that the map (8.1) preserves the sign of the mode number of each operator.

As emphasized for the case of a single cycle of length two in Ref. [3], the reduced Virasoro generators (8.5) at $c_j(\sigma) = 26$ are generically unconventional in form: We remind that the spectral data of each element $\omega(\sigma) \in H'_{26}$ is recorded in the conventional orbifold fraction $n(r)/\rho(\sigma)$. The cycle length $f_j(\sigma)$ in the characteristic ratio $(f_j(\sigma)\frac{n(r)}{\rho(\sigma)})$ seen here represents the effect on $\omega(\sigma)$ due to the unwinding of cycle j in each element of the basic permutation group $H(\text{perm})_K$ of the orbifold-string theories. Beyond the orbifold program and the reduction procedure described here, we are presently unaware of any alternate path to these new Virasoro generators.

Two further remarks are relevant before discussing the reduced form of the extended physical-state condition in the following section. The first remark concerns the *target space-time structure* of these theories, which is *invariant* under the reduction. In particular, each zero mode and hence the space-time dimension of each cycle is unchanged by the map

$$J_j(0)_\sigma = \hat{J}_j(0)_\sigma, \quad \theta\{J_j(0)_\sigma\} = \theta\{\hat{J}_j(0)_\sigma\}, \quad (8.7a)$$

$$D_j(\sigma) = \hat{D}_j(\sigma) \quad (8.7b)$$

and in fact the reduced formulation gives us a slightly more uniform labeling of the momenta and the momentum-squared operator of each cycle:

$$\{J_j(0)_\sigma\} : \quad J_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu(\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)})j}(0), \quad \hat{j}' = 0, 1, \dots, f_j(\sigma) - 1, \quad (8.8a)$$

$$\begin{aligned} P_j^2(\sigma) &= \hat{P}_j^2(\sigma) \\ &= - \sum_{\mu, \nu} \sum_{\hat{j}'=0}^{f_j(\sigma)-1} \theta\{J_j(0)_\sigma\} \mathcal{G}_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu; \frac{-\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \nu}(\sigma) J_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu j}(0) J_{\frac{-\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \nu j}(0). \quad (8.8b) \end{aligned}$$

Here the type I zero modes are included at $\hat{j}' = 0$ (see Eq. (6.5b)), and the form of the momentum-squared operator $\hat{P}_j^2(\sigma)$ at $\hat{c}(\sigma) = 26f_j(\sigma)$ was given in Eq. (6.7b). Similarly,

the reduced number operator $R_j(\sigma)$ in the decomposition of $L_j(0)$ is the same

$$L_j(0) = \frac{1}{2}(-P_j^2(\sigma) + R_j(\sigma)) + \hat{\delta}_{0j}(\sigma), \quad (8.9a)$$

$$\begin{aligned} R_j(\sigma) &= \hat{R}_j(\sigma) \\ &= \left(\sum_{n(r)\mu\nu} \sum_{P \in \mathbb{Z}} \right)' : J_{n(r)\mu j}(P + f_j(\sigma) \frac{n(r)}{\rho(\sigma)}) J_{-n(r),\nu j}(-P - f_j(\sigma) \frac{n(r)}{\rho(\sigma)}) :_M \end{aligned} \quad (8.9b)$$

where the $\hat{c}(\sigma) = 26f_j(\sigma)$ form of $\hat{R}_j(\sigma)$ was given in Eq. (7.4c).

The second remark concerns the twist-field state $|0\rangle_{j\sigma}$ whose definition in Eq. (5.13) translates in the $c_j(\sigma) = 26$ formulation to

$$J_{n(r)\mu j}((M + f_j(\sigma) \frac{n(r)}{\rho(\sigma)}) \geq 0) |0\rangle_{j\sigma} = 0. \quad (8.10)$$

Under the action of the reduced Virasoro generators, we find then that the conformal weights of this state are shifted as follows

$$(L_j(M \geq 0) - \delta_{M,0} \hat{\delta}_{0j}(\sigma)) |0\rangle_{j\sigma} = 0, \quad (8.11a)$$

$$(L_\sigma(M \geq 0) - \delta_{M,0} \hat{\delta}_\sigma) |0\rangle_\sigma = 0, \quad (8.11b)$$

$$|0\rangle_\sigma = \bigotimes_j |0\rangle_{j\sigma}, \quad \hat{\delta}_\sigma \equiv \sum_j \hat{\delta}_{0j}(\sigma) \quad (8.11c)$$

where $L_\sigma(M) = \sum_j L_j(M)$ are the total Virasoro generators of sector σ .

In summary so far, the conformal-field-theoretic shifts we have observed in the reduction

$$\hat{c}_j(\sigma) = 26f_j(\sigma) \longrightarrow c_j(\sigma) = 26, \quad (8.12a)$$

$$\hat{\Delta}_{0j}(\sigma) \longrightarrow \hat{\delta}_{0j}(\sigma), \quad (8.12b)$$

$$j = 0, 1, \dots, N(\sigma) - 1, \quad (8.12c)$$

$$\hat{c}(\sigma) = 26K \longrightarrow c(\sigma) = 26N(\sigma) \quad (8.12d)$$

are generalizations of the (inverse of) the central charge and conformal-weight shifts found in the original orbifold-induction procedure [7] and Ref. [3].

9 Equivalent $c_j(\sigma) = 26$ description of the physical states

There are two interpretations of the map (8.1) and its inverse. In the original conformal-field-theoretic interpretation of Ref. [7], the results of the previous section provide the construction (by relabeling) of one *distinct* CFT in terms of another. We are not directly concerned with this CFT interpretation here. There is however a second interpretation of the reduction

procedure for the orbifold-*string* theories of permutation-type, as restricted by the extended physical-state conditions (7.2) or (7.8) at $\hat{c}_j(\sigma) = 26f_j(\sigma)$. In this string-theoretic interpretation, the map gives us a *completely equivalent* $c_j(\sigma) = 26$ description of the physical spectrum of each cycle j in every sector σ of the new string theories.

Indeed, it is easily checked that all the components $\tilde{j} = 0, 1, \dots, f_j(\sigma) - 1$ of the extended physical-state conditions (7.2) of cycle j map directly onto the simpler physical-state condition of cycle j in the reduced $c_j(\sigma) = 26$ description:

$$(L_j(M \geq 0) - \delta_{M,0}) |\chi(\sigma)\rangle_j = 0, \quad (9.1a)$$

$$j = 0, 1, \dots, N(\sigma) - 1. \quad (9.1b)$$

Here the reduced mode-ordered Virasoro generators $\{L_j(M)\}$ of cycle j are given in Eq. (8.5) and $N(\sigma)$ is the number of cycles in sector σ . Note that the reduced physical state conditions (9.1) are *conventional*, in that they exhibit *unit intercept* for each cycle-string j . We finally emphasize with Ref. [3] that the states described here are *exactly the same physical states* $|\chi(\sigma)\rangle_j$ – now rewritten in terms of the reduced currents – which were originally defined by the extended physical-state conditions (7.2) of the unreduced formulation at $\hat{c}_j(\sigma) = 26f_j(\sigma)$.

For example, assuming again that the zero-modes $\{J_j(0)_\sigma\} = \{\hat{J}_j(0)_\sigma\}$ of cycle j in sector σ are not an empty set, the physical ground-state of cycle j in sector σ is the *same* momentum-boosted twist-field state as determined earlier (see Eq. (7.5)) in the unreduced formulation:

$$|0, J_j(0)\rangle_\sigma = |0, \hat{J}_j(0)\rangle_\sigma, \quad (9.2a)$$

$$J_{n(r)\mu j}((M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)} > 0) |0, J_j(0)\rangle_\sigma = 0, \quad (9.2b)$$

$$L_j(M > 0) |0, J_j(0)\rangle_\sigma = 0, \quad (9.2c)$$

$$P_j^2(\sigma) |0, J_j(0)\rangle_\sigma = P_j^2(\sigma)_{(0)} |0, J_j(0)\rangle_\sigma, \quad (9.2d)$$

$$P_j^2(\sigma)_{(0)} = \hat{P}_j^2(\sigma)_{(0)} = 2(\hat{\delta}_{0j}(\sigma) - 1) \geq -2. \quad (9.2e)$$

Similarly, using the commutator (8.2b), the decomposition (8.9a) and the reduced physical-state conditions (9.1), one finds the level-spacing in the reduced description of cycle j as

$$\Delta(P_j^2(\sigma)) = \Delta(R_j(\sigma)) = 2|M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}|. \quad (9.3)$$

This spacing results when a negatively-moded reduced current $J_{n(r)\mu j}((M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) < 0)$ is added to any previous state $J \dots J |0, J_j(0)\rangle_\sigma$. Recalling that $M = f_j(\sigma)m + \hat{j}$ in the fundamental range of \hat{j} , these increments are recognized as the *same* increments (7.6) obtained in the $\hat{c}_j(\sigma) = 26f_j(\sigma)$ description of the cycle.

Finally a right-mover copy of the reduced physical-state conditions at $c_j(\sigma) = 26$ must be added

$$(L_j^L(M \geq 0) - \delta_{M,0}) |\chi(\sigma)\rangle_j = (L_j^R(M \geq 0) - \delta_{M,0}) |\chi(\sigma)\rangle_j = 0 \quad (9.4)$$

to obtain the reduced description of the physical states of the closed-string sectors of the generalized permutation-orbifolds. Again, these are the *same* physical states described at $\hat{c}_j(\sigma) = 26f_j(\sigma)$ in Eq. (7.8). The reduced system (9.4) decomposes as follows

$$J_j^R(0)_\sigma = J_j^L(0)_\sigma = \frac{1}{\sqrt{2}}J_j(0), \quad (9.5a)$$

$$P_j^2(\sigma)^R = P_j^2(\sigma)^L = \frac{1}{2}P_j^2(\sigma), \quad (9.5b)$$

$$P_j^2(\sigma) |\chi(\sigma)\rangle_j = 2(P_j^2(\sigma)_{(0)} + R_j^L(\sigma)) |\chi(\sigma)\rangle_j, \quad (9.5c)$$

$$(R_j^R(\sigma) - R_j^L(\sigma)) |\chi(\sigma)\rangle_j = 0, \quad (9.5d)$$

$$L_j^R(M > 0) |\chi(\sigma)\rangle_j = L_j^L(M > 0) |\chi(\sigma)\rangle_j = 0, \quad (9.5e)$$

$$P_j^2(\sigma)_{(0)}^{\text{closed}} = \hat{P}_j^2(\sigma)_{(0)}^{\text{closed}} = 4(\hat{\delta}_{0j}(\sigma) - 1) \geq -4 \quad (9.5f)$$

so that the *same* value (7.11) of the ground-state momentum-squared is obtained as well in the reduced formulation of the closed-string sectors.

10 Example: The “pure” permutation orbifolds

The general formulae above are model-dependent, both in regard to $H(\text{perm})_K$ (where we have been explicit), and especially on the choice of element $\omega(\sigma) \in H'_{26}$ (whose information is encoded in the quantities $\{n(r)\mu, \mathcal{G}(\sigma)\}$). These formulae will be used extensively in succeeding papers to study large classes of models with explicit, non-trivial H'_{26} .

In this paper, however, we limit ourselves only to the very simplest orbifold-string theories of permutation-type

$$\left[\frac{U(1)^{26K}}{H(\text{perm})_K} \right]_{\text{open}}, \quad \left[\frac{U(1)^{26K}}{H(\text{perm})_K} \right] \quad (10.1)$$

that is, the closed- and open-string analogues of the “pure” permutation orbifolds with trivial H'_{26} . The sectors σ of these orbifolds are described by the equivalence classes of $H(\text{perm})_K$ alone, and the open-string sectors of the orientation-orbifolds $U(1)^{26}/\mathbb{Z}_2(w.s.)$ are included in the open-string analogues when $K = 2$.

For these cases, the solutions to the H'_{26} eigenvalue problem (2.4a) is very simple:

$$\omega(\sigma) = U(\sigma) = 1, \quad \rho(\sigma) = 1, \quad \bar{n}(0) = 0, \quad (10.2a)$$

$$\mu = a = 0, 1, \dots, 25, \quad (10.2b)$$

$$\mathcal{G}(\sigma) = \mathcal{G}(\sigma) = G_{ab} = -\eta_{ab}, \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad (10.2c)$$

$$\sum_\mu = \sum_a = \dim[\bar{n}(0)] = 26. \quad (10.2d)$$

Here we have chosen the single spectral index in $\{\bar{n}(r)\}$ to be $\bar{n}(0) = 0$, with degeneracy 26 described by the degeneracy index $\mu = a$. The 26-dimensional Minkowski metric η is inherited directly from the untwisted copies of the critical closed-string $U(1)^{26}$.

With the data (10.3) for trivial H'_{26} , the orbifold Virasoro generators and the algebras of twisted sector σ are easily read from Eqs. (4.4), (5.5b) and (5.8):

$$\begin{aligned} \hat{L}_{jj}(m + \frac{j}{f_j(\sigma)}) &= \delta_{m+\frac{j}{f_j(\sigma)},0} \hat{\Delta}_{0j}(\sigma) \\ &\quad - \frac{1}{2f_j(\sigma)} \eta^{ab} \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} : \hat{J}_{0a\hat{\ell}j}(p + \frac{\hat{\ell}}{f_j(\sigma)}) \hat{J}_{0b,\hat{j}-\hat{\ell},j}(m-p + \frac{j-\hat{\ell}}{f_j(\sigma)}) :_M, \end{aligned} \quad (10.3a)$$

$$\hat{\delta}_{0j}(\sigma) = 0, \quad \hat{\Delta}_{0j}(\sigma) = \frac{13}{12} \left(f_j(\sigma) - \frac{1}{f_j(\sigma)} \right), \quad (10.3b)$$

$$\begin{aligned} &[\hat{L}_{jj}(m + \frac{j}{f_j(\sigma)}), \hat{L}_{jj}(n + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ &= \delta_{j\ell} \left\{ (m-n - \frac{j-\hat{\ell}}{f_j(\sigma)}) \hat{L}_{\hat{j}+\hat{\ell},j}(m+n + \frac{j+\hat{\ell}}{f_j(\sigma)}) \right. \\ &\quad \left. + \frac{26f_j(\sigma)}{12} (m + \frac{j}{f_j(\sigma)}) ((m + \frac{j}{f_j(\sigma)})^2 - 1) \delta_{m+n+\frac{j+\hat{\ell}}{f_j(\sigma)},0} \right\}, \end{aligned} \quad (10.3c)$$

$$[\hat{L}_{jj}(m + \frac{j}{f_j(\sigma)}), \hat{J}_{0a\hat{\ell}j}(n + \frac{\hat{\ell}}{f_\ell(\sigma)})] = -\delta_{j\ell} (n + \frac{\hat{\ell}}{f_\ell(\sigma)}) \hat{J}_{0a,\hat{j}+\hat{\ell},j}(m+n + \frac{j+\hat{\ell}}{f_j(\sigma)}), \quad (10.3d)$$

$$[\hat{J}_{0a\hat{j}j}(m + \frac{j}{f_j(\sigma)}), \hat{J}_{0b\hat{\ell}j}(n + \frac{\hat{\ell}}{f_\ell(\sigma)})] = \delta_{j\ell} \eta_{ab} f_j(\sigma) (n + \frac{\hat{\ell}}{f_\ell(\sigma)}) \delta_{m+n+\frac{j+\hat{\ell}}{f_j(\sigma)},0}, \quad (10.3e)$$

$$\begin{aligned} \bar{j} &= 0, 1, \dots, f_j(\sigma) - 1, \quad a = 0, 1, \dots, 25, \\ j &= 0, 1, \dots, N(\sigma) - 1, \quad \sum_j f_j(\sigma) = K. \end{aligned} \quad (10.3f)$$

We remind that $f_j(\sigma)$ is the length of cycle j in sector σ and the summation convention is assumed for repeated indices a, b . To this list, we may add the periodicity conditions

$$\hat{L}_{\hat{j}+f_j(\sigma),j}(m + \frac{j \pm f_j(\sigma)}{f_j(\sigma)}) = \hat{L}_{jj}(m \pm 1 + \frac{j}{f_j(\sigma)}), \quad (10.4a)$$

$$\hat{J}_{0a,\hat{j} \pm f_j(\sigma),j}(m + \frac{j \pm f_j(\sigma)}{f_j(\sigma)}) = \hat{J}_{0a\hat{j}j}(m \pm 1 + \frac{j}{f_j(\sigma)}) \quad (10.4b)$$

and the adjoint operations in these theories

$$\hat{J}_{0a\hat{j}j}(m + \frac{j}{f_j(\sigma)})^\dagger = \hat{J}_{0a,-\hat{j},j}(-m - \frac{j}{f_j(\sigma)}), \quad (10.5a)$$

$$\hat{L}_{\hat{j}j}(m + \frac{j}{f_j(\sigma)})^\dagger = \hat{L}_{-\hat{j},j}(-m - \frac{j}{f_j(\sigma)}), \quad \hat{L}_{0j}(m)^\dagger = \hat{L}_{0j}(-m), \quad (10.5b)$$

$$\|\hat{J}_{0a\hat{\ell}j}((m + \frac{\hat{\ell}}{f_\ell(\sigma)}) < 0) |0, \hat{J}_l(0)\rangle_\sigma\|^2 = G_{aa} f_\ell(\sigma) |m + \frac{\hat{\ell}}{f_\ell(\sigma)}| \|\hat{J}_l(0)\rangle_\sigma\|^2. \quad (10.5c)$$

In fact, Ref. [11] gives a form for the adjoint operation in any current-algebraic orbifold, but we give this result here only for these simple cases. Note in particular that the only negative-norm basis states are associated here to the time-direction $a = 0$ with $G_{00} = -1$.

We also remind that the cycle and sector central charges of these theories are

$$\hat{c}_j(\sigma) = 26f_j(\sigma), \quad \hat{c}(\sigma) = \sum_j \hat{c}_j(\sigma) = 26K. \quad (10.6)$$

The system above is therefore an abelian limit of the results given for the pure WZW permutation orbifolds [15] with $\hat{c}_j(\sigma) = c_g f_j(\sigma)$ and $\hat{c}(\sigma) = K c_g$, where c_g is the central charge of the affine–Sugawara construction [22] on g .

The zero-modes (momenta) of cycle j in sector σ are entirely of type I (see Sec. 6) in all these cases

$$\{\hat{J}_j(0)_\sigma\} = \{\hat{J}_{0a0j}(0), a = 0, 1, \dots, 25\}, \quad (10.7a)$$

$$\hat{D}_j(\sigma) = 26, \quad \hat{D}(\sigma) = \sum_j \hat{D}_j(\sigma) = 26N(\sigma), \quad (10.7b)$$

$$\hat{P}_j^2(\sigma) = \eta^{ab} \hat{J}_{0a0j}(0) \hat{J}_{0b0j}(0), \quad (10.7c)$$

$$\hat{P}_j^2(\sigma)_{(0)} = -2, \quad \Delta(\hat{P}_j^2(\sigma)) = 2f_j(\sigma)|m + \frac{\hat{j}}{f_j(\sigma)}| \quad (10.7d)$$

where $\hat{D}_j(\sigma) = 26$ is the target space-time dimension of cycle j in sector σ and $N(\sigma)$ is the number of cycles in sector σ . We note in particular that each cycle j , though described here at $\hat{c}_j(\sigma) = 26f_j(\sigma)$, has exactly 26 space-time dimensions at ground-state mass-squared -2 , just as in an ordinary untwisted critical open string. Of course, the data given so far is for the open-string sectors of $[U(1)^{26K}/H(\text{perm})_K]_{\text{open}}$, whereas a right-mover copy is needed to describe the closed-string sectors of $U(1)^{26K}/H(\text{perm})_K$. In the latter cases we find that each cycle has 26 left- and right-mover momenta and $\hat{P}_j^2(\sigma)^{\text{closed}} = -4$, again the same as an ordinary untwisted critical closed string. The open- and closed-string forms of the extended physical-state conditions are given respectively in Eqs. (7.2) and (7.8).

Let us turn finally to the equivalent, reduced formulation of these theories at reduced cycle central charge $c_j(\sigma) = 26$, where one finds from Eqs. (8.2),(8.5) and (10.2) that

$$L_j(M) = -\frac{1}{2}\eta^{ab} \sum_{P \in \mathbb{Z}} :J_{0aj}(P) J_{0bj}(M-P):_M, \quad (10.8a)$$

$$[L_j(M), L_\ell(N)] = \delta_{j\ell} \left\{ (M-N)L_j(M+N) + \frac{26}{12}M(M^2-1)\delta_{M+N,0} \right\}, \quad (10.8b)$$

$$[L_j(M), J_{0a\ell}(N)] = -\delta_{j\ell} N J_{0a\ell}(M+N), \quad (10.8c)$$

$$[J_{0aj}(M), J_{0b\ell}(N)] = -\delta_{j\ell} \eta_{ab} N \delta_{M+N,0}, \quad (10.8d)$$

$$a = 0, 1, \dots, 25, \quad j = 0, 1, \dots, N(\sigma) - 1. \quad (10.8e)$$

As discussed more generally in Sec. 8, the zero-modes (momenta) in the reduced formulation are isomorphic to those in the unreduced formulation:

$$\{J_j(0)_\sigma\} = \{\hat{J}_j(0)_\sigma\} = \{J_{0aj}(0), a = 0, 1, \dots, 25\}, \quad (10.9a)$$

$$D_j(\sigma) = \hat{D}_j(\sigma) = 26 = c_j(\sigma), \quad D(\sigma) = \hat{D}(\sigma) = 26N(\sigma) = c(\sigma), \quad (10.9b)$$

$$P_j^2(\sigma) = \hat{P}_j^2(\sigma) = \eta^{ab} J_{0a0j}(0) J_{0b0j}(0), \quad (10.9c)$$

$$P_j^2(\sigma)_{(0)} = \hat{P}_j^2(\sigma)_{(0)} = -2, \quad \Delta P_j^2(\sigma)_{(0)} = \Delta \hat{P}_j^2(\sigma)_{(0)} = 2|M|. \quad (10.9d)$$

The last result is the level-spacing induced by adding an extra negatively-moded current $J_{0aj}(M < 0)$ to a lower-level state and, recalling that $M_j = mf_j(\sigma) + \tilde{j}$, one checks that this level-spacing is indeed the same as that given in Eq. (10.7d) for the unreduced currents.

We end our technical discussion with the adjoints and norms in the reduced formulation

$$J_{0aj}(M)^\dagger = J_{0aj}(-M), \quad L_j(M)^\dagger = L_j(-M), \quad (10.10a)$$

$$\|J_{0a\ell}(M < 0) |0, J_l(0)_\sigma\rangle\|^2 = G_{aa}|M| \| |0, J_l(0)_\sigma\rangle\|^2 \quad (10.10b)$$

where the adjoints are the map of Eqs. (10.5a,b) and the norms are computed from the adjoints. Again using $M_\ell = f_\ell(\sigma)m + \tilde{\ell}$, we see that the norms are the same as those computed in Eq. (10.5c) for the original formulation. Similarly, of course, the inner product of any two states are the same in the reduced and unreduced formulations.

Taken together with the reduced physical-state conditions (9.1) and (9.4) for the open- and closed-string sectors, these results allow us to conclude the following on inspection: Each cycle j of each sector σ of the “pure” orbifold-strings (10.1) is nothing but an ordinary untwisted 26-dimensional string with target space-time symmetry $\text{SO}(25, 1)$. This conclusion is however quite special for the “pure” orbifold-string systems with trivial H'_{26} , whereas (as seen for $H(\text{perm})_2 = \mathbb{Z}_2$ and $\mathbb{Z}_2(\text{w.s.})$ in Ref. [3]) the orbifold-string theories with non-trivial H'_{26} are generically new.

The critical-string equivalences of this section were anticipated for the “pure” orbifolds of permutation-type with $H(\text{perm})_2 = \mathbb{Z}_2$ or $\mathbb{Z}_2(\text{w.s.})$ in Ref. [3], and were verified at the interacting level [5] for the pure permutation orbifolds with $H(\text{perm})_K = \mathbb{Z}_K$, K prime. Moreover, our conclusion here was conjectured for all $H(\text{perm})_K$ in Ref. [5]. It should be added that special cases with particular non-trivial H'_{26} (see e.g. the orientation-orbifold string system in Ref. [4]) can also be equivalent to ordinary critical strings, including the critical bosonic open-closed string system. Taken together then, the orbifold-string systems of permutation-type provide several rising, ever-more twisted hierarchies of new string theories, including ordinary critical strings as the simplest cases.

We finally note that the pure permutation orbifolds

$$\frac{\text{U}(1)^{26K}}{H(\text{perm})_K}, \quad (10.11)$$

being composed entirely of ordinary closed-string sectors, exhibit *multiple gravitons*. Indeed, we have seen here that the free theories in these cases exhibit one graviton per cycle per sector. The only examples studied so far at the symmetrized, interacting level are the prime cyclic permutation orbifolds

$$\frac{\text{U}(1)^{26\lambda}}{\mathbb{Z}_\lambda}, \quad \lambda \text{ prime} \quad (10.12)$$

where the linear (diagonal) modular-invariant construction of Ref. [5] shows the total number of gravitons

$$N_\lambda = \lambda + (\lambda - 1) = 2\lambda - 1. \quad (10.13)$$

This includes in particular one graviton in each nontrivial twisted sector. The interaction (or presumably non-interaction) among these gravitons will require the construction of the twist fields (intertwiners) among the sectors, an inquiry which is beyond the scope of this paper. We similarly expect more than one graviton in the generalized permutation orbifolds with nontrivial H'_{26} , at least from the cycles of the sector corresponding to the unit elements of $H(\text{perm})_K$ and H'_{26} . On the other hand – as we will discuss in succeeding papers – the open-closed string systems of the orientation-orbifolds [3,4]

$$\frac{\text{U}(1)^{26K}}{H_-} = \frac{\text{U}(1)_L^{26} \times \text{U}(1)_R^{26}}{H_-}, \quad H_- \subset \mathbb{Z}_2(\text{w.s.}) \times H'_{26} \quad (10.14)$$

have only a single graviton for any choice of H'_{26} .

11 Conclusions

In the previous paper [6] of this series, we used BRST quantization to find the orbifold Virasoro algebras and extended physical-state conditions of the bosonic prototypes of the orbifold-string theories of permutation-type:

$$\frac{\text{U}(1)^{26K}}{H_+}, \quad \left[\frac{\text{U}(1)^{26K}}{H_+} \right]_{\text{open}}, \quad H_+ \subset H(\text{perm})_K \times H'_{26} \quad (11.1a)$$

$$\frac{\text{U}(1)^{26}}{H_-} = \frac{\text{U}(1)_L^{26} \times \text{U}(1)_R^{26}}{H_-}, \quad H_- \subset \mathbb{Z}_2(\text{w.s.}) \times H'_{26}. \quad (11.1b)$$

These theories live at cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$, where $f_j(\sigma)$ is the length of cycle j in each equivalence class σ of the permutation group $H(\text{perm})_K$ or $\mathbb{Z}_2(\text{w.s.})$. The expected sector central charges $\hat{c}(\sigma) = 26K$ of each orbifold are obtained by summing over the cycles of sector σ .

In this paper we have completed the cycle dynamics of these theories, supplementing the extended physical-state conditions of cycle j with the explicit form of the orbifold Virasoro generators as functions of the twisted matter of each cycle. Our results here are general, depending on the choice of element $\omega(\sigma) \in H'_{26}$ in the divisors of each orbifold. With these tools, we also began a systematic inquiry into the target space-time structure of these theories, including in particular the number $\hat{D}_j(\sigma)$ of target space-time dimensions in cycle j of sector σ .

We also found an equivalent, reduced description of the physical states of each cycle at reduced cycle central charge $c_j(\sigma) = 26$, emphasizing that the target space-time properties of the theories are invariant under the reduction and in fact more transparent in the reduced formulation.

As examples, the simplest cases with trivial H'_{26} (the orbifolds of “pure” permutation-type)

$$\frac{\mathrm{U}(1)^{26K}}{H(\mathrm{perm})_K}, \quad \left[\frac{\mathrm{U}(1)^{26K}}{H(\mathrm{perm})_K} \right]_{\mathrm{open}}, \quad \frac{\mathrm{U}(1)^{26}}{\mathbb{Z}_2(\mathrm{w.s.})} \quad (11.2)$$

were worked out in some detail, with the result that $\hat{D}_j(\sigma) = 26$ for each cycle j in each sector σ of these examples. Indeed, the reduced formulation transparently shows that each of these cycles is spectrally equivalent to an ordinary untwisted 26-dimensional string.

This is not the case however for the more general situation with nontrivial H'_{26} , which provides large classes of new string theories. In particular, it is clear from our discussion that the target space-time dimensionality of these theories is not generically equal to any of the central charges discussed above.

In the following paper, we will apply the general formulae developed here to study a large example of non-trivial H'_{26} , finding that the new string theories in fact exhibit many target space-times of varying dimensionality, symmetry and signature – including in particular Lorentzian target space-times with $\hat{D}_j(\sigma) \leq 26$.

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The orbifold-string theories of permutation-type: II. Cycle dynamics and target space-time dimensions

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Abstract

We continue our discussion of the general bosonic prototype of the new orbifold-string theories of permutation-type. Supplementing the extended physical-state conditions of the previous paper, we construct here the extended Virasoro generators with cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$, where $f_j(\sigma)$ is the length of cycle j in twisted sector σ . We also find an equivalent, reduced formulation of each physical-state problem at reduced cycle central charge $c_j(\sigma) = 26$. These tools are used to begin the study of the target space-time dimension $\hat{D}_j(\sigma)$ of cycle j in sector σ , which is naturally defined as the number of zero modes (momenta) of each cycle. The general model-dependent formulae derived here will be used extensively in succeeding papers, but are evaluated in this paper only for the simplest case of the “pure” permutation orbifolds.

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1 Introduction

The new orbifold-string theories of permutation-type [1-6] include the bosonic prototypes

$$\frac{U(1)^{26K}}{H_+} = \frac{U(1)_1^{26} \times \cdots \times U(1)_K^{26}}{H_+}, \quad H_+ \subset H(\text{perm})_K \times H'_{26}, \quad (1.1a)$$

$$\left[\frac{U(1)^{26K}}{H_+} \right]_{\text{open}}, \quad (1.1b)$$

$$\frac{U(1)^{26K}}{H_-} = \frac{U(1)_L^{26} \times U(1)_R^{26}}{H_-}, \quad H_- \subset \mathbb{Z}_2(\text{w.s.}) \times H'_{26} \quad (1.1c)$$

and generalizations of these, as noted in Appendix B of Ref. 6. The three families in (1.1) are called respectively the generalized permutation orbifolds (twisted closed strings at sector central charge $\hat{c} = 26K$), the open-string analogues of the generalized permutation orbifolds (twisted open strings at $\hat{c} = 26K$), and the orientation-orbifold string systems, which contain an equal number of twisted closed strings at $\hat{c} = 26$ and twisted open strings at $\hat{c} = 52$. The open-string sectors of the orientation orbifolds are contained, along with their T -duals, at $K = 2$ in the open-string analogues of the generalized permutation orbifolds. The closed-string sectors of the orientation orbifolds form the ordinary space-time orbifold $U(1)^{26}/H'_{26}$ at $\hat{c} = 26$. Further information on special cases of these orbifold-string systems, especially $\hat{c} = 52$, is contained in Refs. [3-5]. We note in particular that the orientation-orbifold string systems (1.1c) generalize and include [4] the ordinary critical bosonic open-closed string system.

In the previous paper [6] of the present series, cycle-bases of general permutation groups and the principles of the orbifold program [7-21] were used to construct a twisted BRST system for each cycle j in each twisted sector σ of these orbifolds, including the extended algebra of the BRST charges

$$[\hat{Q}_i(\sigma), \hat{Q}_j(\sigma)]_+ = 0 \quad \forall i, j \text{ in sector } \sigma, \quad (1.2)$$

and right-mover copies of these systems in the twisted closed-string sectors. Moreover, the BRST systems were used to find the *extended physical-state conditions* of the matter in

cycle j of sector σ

$$(\hat{L}_{jj}((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) - \hat{a}_{f_j(\sigma)} \delta_{m + \frac{\hat{j}}{f_j(\sigma)}, 0}) |\chi(\sigma)\rangle_j = 0, \quad (1.3a)$$

$$\begin{aligned} & [\hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{L}_{\ell\ell}(m + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ &= \delta_{j\ell} \left\{ (m - n - \frac{\hat{j} - \hat{\ell}}{f_j(\sigma)}) \hat{L}_{\hat{j} + \hat{\ell}, j}(m + n + \frac{\hat{j} + \hat{\ell}}{f_j(\sigma)}) \right. \\ & \quad \left. + \frac{1}{12} \hat{c}_j(\sigma) (m + \frac{\hat{j}}{f_j(\sigma)}) ((m + \frac{\hat{j}}{f_j(\sigma)})^2 - 1) \delta_{m + n + \frac{\hat{j} + \hat{\ell}}{f_j(\sigma)}, 0} \right\}, \end{aligned} \quad (1.3b)$$

$$\hat{c}_j(\sigma) = 26f_j(\sigma), \quad \hat{a}_{f_j(\sigma)} = \frac{13f_j^2(\sigma) - 1}{12f_j(\sigma)}, \quad (1.3c)$$

$$\bar{j} = 0, 1, \dots, f_j(\sigma) - 1, \quad j = 0, 1, \dots, N(\sigma) - 1, \quad \sum_j f_j(\sigma) = K \quad (1.3d)$$

including again a right-mover copy of these conditions for twisted closed-string sectors. The algebra (1.3b) of the matter generators $\{\hat{L}_{jj}\}$ is called the *orbifold Virasoro algebra* [7,15,6] of cycle j in sector σ . The fundamental numbers (1.3c) of each cycle are the *cycle central charge* $\hat{c}_j(\sigma)$ and the *cycle-intercept* $\hat{a}_{f_j(\sigma)}$, both expressed in terms of the length $f_j(\sigma)$ of cycle j in sector σ . Using the final sum rule in Eq. (1.3d) the reader easily verifies that the *sector central charges*

$$\hat{c}(\sigma) = \sum_j \hat{c}_j(\sigma) \quad (1.4)$$

are $26K$ for the closed- and open-string counterparts of the generalized permutation orbifolds and 52 ($K = f_0(\sigma) = 2$) for the twisted open-string sectors of the orientation orbifolds. The twisted closed-string sectors of the orientation orbifolds (the ordinary space-time orbifold $U(1)^{26}/H'_{26}$) can also be obtained from these results by choosing $K = N(\sigma) = f_0(\sigma) = 1$ and hence the ordinary values $\hat{a}_1 = 1, \hat{c}(\sigma) = \hat{c}_0(\sigma) = 26$.

In fact these results see only the permutation subgroup $H(\text{perm})_K$ or $\mathbb{Z}_2(\text{w.s.})$ of H_\pm , which determines the twisted permutation gravities [2] of each sector and hence the BRST systems. The 26-dimensional automorphism subgroup H'_{26} of H_\pm , which operates uniformly on each left- and right-mover copy of the critical closed string, is encoded however in the explicit form of the extended Virasoro generators of the matter.

Our first task in this paper is therefore to supplement the extended physical-state conditions (1.3) with the construction of the orbifold Virasoro generators $\{\hat{L}_{jj}\}$ at cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$ as functions of the twisted matter. This construction encodes the solution of the spectral problem of each element $\omega(\sigma) \in H'_{26}$ of the 26-dimensional automorphism subgroup, and our general formulae will be evaluated explicitly for a large class of examples of H'_{26} in the following paper. Subexamples of this construction at $\hat{c}(\sigma) = 52$ and $\hat{c}(\sigma) = 26\lambda, \lambda$ prime have already been discussed in Refs. [3-5].

Generalizing our work at $\hat{c}(\sigma) = 52$ in Ref. [3], we shall also find an equivalent, *reduced* form of the physical-state problem for each cycle j of each sector σ at *reduced* cycle central charge $c_j(\sigma) = 26$.

With these tools, we shall begin a survey of the *space-time (target-space) interpretation* of the orbifold-string theories, noting in particular with Ref. [3] that the target space-times are invariant under the reduction. In this discussion, we will focus on the *target space-time dimension*

$$\hat{D}_j(\sigma) \equiv \dim\{\hat{J}_j(0)_\sigma\} \quad (1.5)$$

of cycle j in sector σ as the number of zero modes (momenta) of the cycle, and following Refs. [3-5], we will define the momentum-squared operators and level-spacing which are needed to analyze the extended physical-state problems. In the only explicit examples of this paper, we shall find that $\hat{D}_j(\sigma) = 26$ for the “pure” permutation orbifolds (with trivial H'_{26}), so that these simple cases are equivalent to collections of ordinary critical strings. More generally however *the dimensionality of the target space-time is not equal to any of the central charges of the theories*, and in succeeding papers we will present many examples of non-trivial H'_{26} with $\hat{D}_j(\sigma) \leq 26$!

2 An application of the orbifold program

To obtain the general forms of the orbifold Virasoro generators for each cycle j of each sector σ , we apply the standard methods of the orbifold program [7-21] which emphasizes the principle of local isomorphisms [9,11,12,15-17].

The orbifold program always begins with the operator-product formulation of the untwisted systems in question. Here we need then only the operator-product form of the stress-tensor/current system of K copies of the critical bosonic string:

$$T_I(z) = \frac{1}{2}G^{ab} :J_{aI}(z)J_{bI}(z):, \quad I = 0, 1, \dots, K-1, \quad (2.1a)$$

$$G = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad a, b = 0, 1, \dots, 25, \quad (2.1b)$$

$$\begin{aligned} T_I(z)T_J(w) = \delta_{IJ} \left(\frac{26/2}{(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) T_I(w) \right) \\ + :T_I(z)T_J(w): \end{aligned} \quad (2.1c)$$

$$\begin{aligned} T_I(z)J_{aJ}(w) = \delta_{IJ} \left(\frac{1}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) J_{aI}(w) \\ + :T_I(z)J_{aJ}(w): \end{aligned} \quad (2.1d)$$

$$J_{aI}(z)J_{bJ}(w) = \frac{\delta_{IJ}G_{ab}}{(z-w)^2} + :J_{aI}(z)J_{bJ}(w):. \quad (2.1e)$$

The symbol $:\cdots:$ is operator-product normal ordering, that is, the operator product minus the singularities shown. The total stress tensor $T(z) = \sum_I T_I(z)$ is also Virasoro with total central charge $c = 26K$, and a right-mover copy of this system is included implicitly in our application below.

The automorphic responses of these operators are

$$J_{aI}(z)' = \omega(\sigma)_a^b W(\sigma)_I^J J_{bJ}(z), \quad (2.2a)$$

$$T_I(z)' = W(\sigma)_I^J T_J(z), \quad (2.2b)$$

$$W(\sigma) \in H(\text{perm})_K, \quad \omega(\sigma) \in H'_{26}. \quad (2.2c)$$

Note that the definition of a sector σ requires the choice of one element $(W \times \omega)$ from each equivalence class of both $H(\text{perm})_K$ and H'_{26} . Here we are following the sector-labeling convention of the orbifold program for product groups, but we mention that σ can equivalently be viewed as a two-component vector, with one component each for $H(\text{perm})_K$ and H'_{26} .

The next step in the orbifold program is to find the so-called eigenfields [9,11,12,15-17] under the automorphism groups, and for this we must first recall the *H-eigenvalue problems* of the group elements: For each element $W(\sigma) \in H(\text{perm})_K$ we have the spectral problem

$$W(\sigma)_I^J V^\dagger(\sigma)_J^{\hat{j}j} = V^\dagger(\sigma)_I^{\hat{j}j} e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}}, \quad (2.3a)$$

$$\bar{j} = 0, 1, \dots, f_j(\sigma) - 1, \quad j = 0, 1, \dots, N(\sigma) - 1, \quad (2.3b)$$

$$\sum_j = N(\sigma), \quad \sum_j f_j(\sigma) = K, \quad (2.3c)$$

where j labels cycles of length $f_j(\sigma)$, $N(\sigma)$ is the number of cycles in $W(\sigma)$, and \hat{j} indexes within each cycle j . The explicit form of the unitary eigenmatrix $V(\sigma)$ is given in Refs. [13,15], and this eigenvalue problem was also discussed in the previous paper [6] on the general BRST problem. What is essential to add here is the eigenvalue problem for each element $\omega(\sigma) \in H'_{26}$ of the 26-dimensional automorphism subgroup:

$$\omega(\sigma)_a^b U^\dagger(\sigma)_b^{n(r)\mu} = U^\dagger(\sigma)_a^{n(r)\mu} e^{-2\pi i \frac{n(r)}{\rho(\sigma)}} \quad (2.4a)$$

$$\bar{n}(r) \in \{0, 1, \dots, \rho(\sigma) - 1\}, \quad \sum_\mu = \dim[\bar{n}(r)], \quad \sum_r \dim[\bar{n}(r)] = 26. \quad (2.4b)$$

Here $U(\sigma)$ is the unitary eigenmatrix of $\omega(\sigma)$, with order $\rho(\sigma)$, and $n(r)$, $\mu = \mu(n(r))$ are respectively the spectral and degeneracy indices of $\omega(\sigma)$. The barred quantities in (2.3b) and (2.4b) are the pullbacks of the spectral indices to their fundamental ranges. Many of these spectral problems [9,11,13,15,16] have been solved explicitly in the orbifold program, but we will not choose any particular non-trivial H'_{26} in the general discussion of this paper (see however Sec. 10).

Given the forms of these two eigenvalue problems, we may write down the *eigenfields* for each $W(\sigma) \in H(\text{perm})_K$ and $\omega(\sigma) \in H'_{26}$:

$$\Theta_{\hat{j}j}(z, \sigma) \equiv \sqrt{f_j(\sigma)} V(\sigma)_{\hat{j}j}^I T_I(z), \quad (2.5a)$$

$$\mathcal{J}_{n(r)\mu\hat{j}j}(z, \sigma) \equiv \chi_{n(r)\mu}(\sigma) \sqrt{f_j(\sigma)} U(\sigma)_{n(r)\mu}^a V(\sigma)_{\hat{j}j}^I J_{aI}(z). \quad (2.5b)$$

Here we have chosen the standard normalization $\chi_{\hat{j}j}(\sigma) = \sqrt{f_j(\sigma)}$ for elements of $H(\text{perm})_K$, but left the normalizations $\chi_{n(r)\mu}(\sigma)$ arbitrary for elements of H'_{26} . The eigenfields are constructed to diagonalize the automorphic responses as follows:

$$\Theta_{\hat{j}j}(z, \sigma)' = e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}} \Theta_{\hat{j}j}(z, \sigma), \quad (2.6a)$$

$$\mathcal{J}_{n(r)\mu\hat{j}j}(z, \sigma)' = e^{-2\pi i (\frac{\hat{j}}{f_j(\sigma)} + \frac{n(r)}{\rho(\sigma)})} \mathcal{J}_{n(r)\mu\hat{j}j}(z, \sigma). \quad (2.6b)$$

Moreover, the eigenfields inherit the following periodicity conditions

$$\Theta_{\hat{j} \pm f_j(\sigma), j}(z, \sigma) = \theta_{\hat{j}j}(z, \sigma), \quad (2.7a)$$

$$\mathcal{J}_{n(r) \pm \rho(\sigma), \mu \hat{j}j}(z, \sigma) = \mathcal{J}_{n(r)\mu, \hat{j} \pm f_j(\sigma), j}(z, \sigma) = \mathcal{J}_{n(r)\mu\hat{j}j}(z, \sigma) \quad (2.7b)$$

from the natural periodicities of the eigenvalue problems.

The composite form and operator products of the eigenfields in terms of themselves are then straightforwardly computed from their definitions and the original operator products (2.1). We will not write them out explicitly here (see however the remark after Eq. (3.4)), but call attention only to some useful quantities, the *twisted metrics*, which appear in the operator products of the eigenfields:

$$\begin{aligned} \mathcal{G}_{\hat{j}j; \hat{\ell}\ell}(\sigma) &= \sqrt{f_j(\sigma)} \sqrt{f_\ell(\sigma)} V(\sigma)_{\hat{j}j}^I V(\sigma)_{\hat{\ell}\ell}^J \delta_{IJ} \\ &= \delta_{j\ell} f_j(\sigma) \delta_{\hat{j} + \hat{\ell}, 0 \bmod f_j(\sigma)}, \end{aligned} \quad (2.8a)$$

$$\begin{aligned} \mathcal{G}_{n(r)\mu; n(s)\nu}(\sigma) &= \chi_{n(r)\mu}(\sigma) \chi_{n(s)\nu}(\sigma) U(\sigma)_{n(r)\mu}^a U(\sigma)_{n(s)\nu}^b G_{ab} \\ &= \delta_{n(r)+n(s), 0 \bmod f_j(\sigma)} \mathcal{G}_{n(r)\mu; -n(r)\nu}(\sigma) \end{aligned} \quad (2.8b)$$

$$\sum_{n(t), \eta} \mathcal{G}^{n(r)\mu; n(t)\eta}(\sigma) \mathcal{G}_{n(t)\eta; n(s)\nu}(\sigma) = \delta_{n(r)\mu}^{n(s)\nu} \quad (2.8c)$$

In what follows, the information about the choice of $\omega(\sigma) \in H'_{26}$ is encoded in the quantities $n(r)\mu$, $\rho(\sigma)$ and the twisted metric $\mathcal{G}(\sigma)$ and its inverse $\mathcal{G}'(\sigma)$ in Eqs. (2.8b,c). The inverse of the metric (2.8a) is obtained by inverting the factor $f_j(\sigma)$, while the inverse of Eq. (2.8b) involves the inverse of the normalizations and replacement of the eigenmatrices by their adjoints.

At this stage we have only rearranged the untwisted theory in terms of the eigenfields $\mathcal{A}(z, \sigma)$. The final step in the orbifold program is the transition to twisted sector σ of the orbifold by an application of the *principle of local isomorphisms* [9,11,12,15-17]

$$\mathcal{A}(z, \sigma) \rightarrow \hat{A}(z, \sigma), \quad (2.9a)$$

$$\text{operator products of } \{\mathcal{A}(z, \sigma)\} \rightarrow \text{operator products of } \{\hat{A}(z, \sigma)\}, \quad (2.9b)$$

$$\text{diagonal automorphic responses} \rightarrow \text{monodromies}, \quad (2.9c)$$

where $\{\hat{A}(z, \sigma)\}$ are now the twisted fields of twisted sector σ of the orbifold. The name of the principle derives from part (b) of Eq. (2.9), that the operator products of the twisted fields are the same as (locally isomorphic to) the operator products of the eigenfields. (We remind that there is another, equivalent way around the commuting diagrams of Refs. [9,11,17] to get from the untwisted fields to the twisted fields. This path involves first a parallel application of the principle of local isomorphisms, followed by a monodromy decomposition to obtain the twisted fields \hat{A} .)

3 The twisted operator products of sector σ

Having completed the steps above, we emerge in the orbifold with the following twisted stress tensors of sector σ

$$\hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) = \sum_{n(r)\mu\nu} \frac{\mathcal{G}^{n(r)\mu; -n(r)\nu}(\sigma)}{2f_j(\sigma)} \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} : \hat{J}_{n(r)\mu\hat{\ell}\hat{j}}(z, \sigma) \hat{J}_{-n(r), \nu, \hat{j}-\hat{\ell}, \hat{j}}(z, \sigma) : \quad (3.1)$$

where $: \dots :$ is now operator-product normal ordering in the orbifold. The monodromies of these operators are

$$\hat{\theta}_{\hat{j}\hat{j}}(ze^{2\pi i}, \sigma) = e^{-2\pi i \frac{\hat{j}}{f_j(\sigma)}} \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma), \quad (3.2a)$$

$$\hat{J}_{n(r)\mu\hat{j}\hat{j}}(ze^{2\pi i}, \sigma) = e^{-2\pi i (\frac{\hat{j}}{f_j(\sigma)} + \frac{n(r)}{\rho(\sigma)})} \hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) \quad (3.2b)$$

and the periodicities

$$\hat{\theta}_{\hat{j}\pm f_j(\sigma), \hat{j}}(z, \sigma) = \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma), \quad (3.3a)$$

$$\hat{J}_{n(r)\pm \rho(\sigma), \mu\hat{j}\hat{j}}(z, \sigma) = \hat{J}_{n(r)\mu, \hat{j}\pm f_j(\sigma), \hat{j}}(z, \sigma) = \hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) \quad (3.3b)$$

are inherited from the eigenfields.

The operator products of sector σ are obtained as

$$\begin{aligned} \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) \hat{\theta}_{\hat{\ell}\hat{\ell}}(\omega, \sigma) &= \delta_{j\ell} \left[\frac{\delta_{\hat{j}+\hat{\ell}, 0 \bmod f_j(\sigma)} \frac{26}{2} f_j(\sigma)}{(z-\omega)^4} \right. \\ &\quad \left. + \left(\frac{2}{(z-\omega^2)^2} + \frac{1}{z-\omega} \partial_\omega \right) \hat{\theta}_{\hat{j}+\hat{\ell}, j}(\omega, \sigma) \right] \\ &\quad + : \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) \hat{\theta}_{\hat{\ell}\hat{\ell}}(\omega, \sigma) : \end{aligned} \quad (3.4a)$$

$$\begin{aligned} \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) \hat{J}_{n(r)\mu\hat{\ell}\hat{\ell}}(\omega, \sigma) &= \delta_{j\ell} \left(\frac{1}{(z-\omega)^2} + \frac{1}{z-\omega} \partial_\omega \right) \hat{J}_{n(r)\mu, \hat{j}+\hat{\ell}, j}(\omega, \sigma) \\ &\quad + : \hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) \hat{J}_{n(r)\mu\hat{\ell}\hat{\ell}}(\omega, \sigma) : \end{aligned} \quad (3.4b)$$

$$\begin{aligned} \hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\hat{\ell}}(\omega, \sigma) &= \delta_{j\ell} \left(\frac{f_j(\sigma) \mathcal{G}_{n(r)\mu; n(s)\nu}(\sigma) \delta_{\hat{j}+\hat{\ell}, 0 \bmod f_j(\sigma)}}{(z-\omega)^2} \right) \\ &\quad + : \hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\hat{\ell}}(\omega, \sigma) : . \end{aligned} \quad (3.4c)$$

The relations above provide a complete description of twisted sector σ . If desired, the previously-omitted details of the eigenfield system can be obtained from these statements by going backward $\hat{A} \rightarrow \mathcal{A}$, while replacing the monodromies (3.2) with the diagonal automorphic responses (2.6) of the eigenfields. We will comment on applications to specific orbifold-string systems after finding the corresponding mode algebras below.

4 The twisted mode algebras of sector σ

The twisted operator-product form of the system above is straightforwardly translated to the mode-algebraic description of the sector. With attention to the monodromies (3.2) and the conformal-weight terms ($\Delta/(z-\omega)^2$) in the operator products, we define the modes of the stress tensors and twisted currents as follows:

$$\hat{\theta}_{\hat{j}\hat{j}}(z, \sigma) = \sum_{m \in \mathbb{Z}} \hat{L}_{\hat{j}\hat{j}}(m + \frac{\hat{j}}{f_j(\sigma)}) z^{-(m + \frac{\hat{j}}{f_j(\sigma)}) - 2}, \quad (4.1a)$$

$$\hat{J}_{n(r)\mu\hat{j}\hat{j}}(z, \sigma) = \sum_{m \in \mathbb{Z}} \hat{J}_{n(r)\mu\hat{j}\hat{j}}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) z^{-(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) - 1}. \quad (4.1b)$$

This gives immediately the mode form of the orbifold Virasoro generators

$$\begin{aligned} \hat{L}_{\hat{j}\hat{j}}(m + \frac{\hat{j}}{f_j(\sigma)}) &= \sum_{n(r)\mu\nu} \frac{\mathcal{G}_{n(r)\mu; -n(r), \nu}(\sigma)}{2f_j(\sigma)} \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} \times \\ &\quad \times : \hat{J}_{n(r)\mu\hat{\ell}\hat{j}}(p + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_j(\sigma)}) \hat{J}_{-n(r), \nu, \hat{j}-\hat{\ell}, j}(m - p - \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}-\hat{\ell}}{f_j(\sigma)}) : \end{aligned} \quad (4.2)$$

and the mode periodicities

$$\hat{L}_{\hat{j}\pm f_j(\sigma),j}(m + \frac{\hat{j}\pm f_j(\sigma)}{f_j(\sigma)}) = \hat{L}_{\hat{j}j}(m \pm 1 + \frac{\hat{j}}{f_j(\sigma)}), \quad (4.3a)$$

$$\begin{aligned} \hat{J}_{n(r)\pm\rho(\sigma),\mu\hat{j}j}(m + \frac{n(r)\pm\rho(\sigma)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) &= \hat{J}_{n(r)\mu,\hat{j}\pm f_j(\sigma),j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}\pm f_j(\sigma)}{f_j(\sigma)}) \\ &= \hat{J}_{n(r)\mu\hat{j}j}(m \pm 1 + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}). \end{aligned} \quad (4.3b)$$

The operator-product normal-ordered forms (4.2) of the orbifold Virasoro generators are not as useful as the mode-normal ordered forms we shall obtain for these generators later.

We give next the twisted mode algebras of sector σ

$$\begin{aligned} &[\hat{L}_{\hat{j}j}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{L}_{\hat{\ell}\ell}(n + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ &= \delta_{j\ell} \left\{ (m - n + \frac{\hat{j}-\hat{\ell}}{f_j(\sigma)}) \hat{L}_{\hat{j}-\hat{\ell},j}(m + n + \frac{\hat{j}+\hat{\ell}}{f_j(\sigma)}) \right. \\ &\quad \left. + \frac{26f_j(\sigma)}{12} (m + \frac{\hat{j}}{f_j(\sigma)}) ((m + \frac{\hat{j}}{f_j(\sigma)})^2 - 1) \delta_{m+n+\frac{\hat{j}+\hat{\ell}}{f_j(\sigma)},0} \right\}, \end{aligned} \quad (4.4a)$$

$$\begin{aligned} &[\hat{L}_{\hat{j}j}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{J}_{n(r)\mu\hat{\ell}\ell}(n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ &= -\delta_{j\ell} (n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}) \hat{J}_{n(r)\mu,\hat{j}+\hat{\ell},j}(m + n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}+\hat{\ell}}{f_j(\sigma)}), \end{aligned} \quad (4.4b)$$

$$\begin{aligned} &[\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}), \hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(s)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ &= \delta_{j\ell} f_j(\sigma) (m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \delta_{n(r)+n(s),0 \bmod \rho(\sigma)} \\ &\quad \times \delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)}+\frac{\hat{j}+\hat{\ell}}{f_j(\sigma)},0} \mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma), \end{aligned} \quad (4.4c)$$

which are obtained by standard [11] orbifold contour methods from the twisted operator products (3.4) and the mode expansions (4.1) of the operators. The reader will recognize in particular the general orbifold Virasoro algebra (4.4a) obtained earlier¹ in Ref. [6] and quoted in Eq. (1.3) of the Introduction. We remind the reader of the ranges given in Eqs. (2.3),(2.4) for the quantum numbers $\hat{j}j$ ($H(\text{perm})_K$) and $n(r)\mu$ (H'_{26}) which appear in this result, as well as the definition of the twisted metric $\mathcal{G}(\sigma)$ in Eq. (2.8b).

In further detail, the orbifold Virasoro algebra (4.4a) of sector σ is semisimple with respect to the cycles j of each $W(\sigma) \in H(\text{perm})_K$, and each cycle has its own integral Virasoro subalgebra at cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$:

$$\begin{aligned} [\hat{L}_{0j}(m), \hat{L}_{0\ell}(n)] &= \delta_{j\ell} \left\{ (m - n) \hat{L}_{0j}(m + n) + \frac{26f_j(\sigma)}{12} m(m^2 - 1) \delta_{m+n,0} \right\}, \\ j, \ell &= 0, 1, \dots, N(\sigma) - 1, \end{aligned} \quad (4.5)$$

¹The general orbifold Virasoro algebra [4.4a] was first obtained in the WZW permutation orbifolds [15] with $26 \rightarrow c_g$, where c_g is the central charge of the affine-Sugawara construction [22] on Lie g .

where $N(\sigma)$ is the number of cycles in sector σ . The total Virasoro generators of sector σ are obtained by summing over the cycles of the sector

$$\hat{L}_\sigma(m) = \sum_j \hat{L}_{0j}(m), \quad \hat{c}(\sigma) = \sum_j \hat{c}_j(\sigma) = 26K \quad (4.6a)$$

$$[\hat{L}_\sigma(m), \hat{L}_\sigma(n)] = (m - n)\hat{L}_\sigma(m + n) + \frac{26K}{12}m(m^2 + 1)\delta_{m+n,0}, \quad (4.6b)$$

where we have used the cycle sum rule in Eq. (2.3c) to obtain the sector central charges $\hat{c}(\sigma)$.

Together, the twisted mode algebras (4.4) and the extended physical-state conditions (1.3a) form what we will call the *general cycle dynamics* of the matter in cycle j of sector σ . (The cycle dynamics includes the composite structure (4.2) of the orbifold Virasoro generators, but we remind that a more useful form of this structure will be obtained in the following section.)

We conclude with some comments on the applicability of the general cycle dynamics to the sectors of the three families (1.1) of orbifold-string theories of permutation-type. These results are complete as they stand for the twisted open string systems, including the open-string analogues (1.1b) of the generalized permutation orbifolds. The open-string sectors of the orientation orbifolds (1.1c) are included in the special case $K = f_0(\sigma) = 2$ and $a_2 = 17/8$ of these results at $\hat{c}(\sigma) = 52$. Right-mover copies of the cycle dynamics must be added to describe the generalized permutation orbifolds in Eq. (1.1a). The cycle dynamics of both open- and closed-string sectors at $\hat{c}(\sigma) = 52$ were described earlier in Ref. [3]. Finally, the cycle dynamics of the closed-string sectors of the orientation orbifolds (the ordinary space-time orbifolds $U(1)^{26}/H'_{26}$) are obtained with a right-mover copy by choosing $K = f_0(\sigma) = 1$ and therefore the conventional intercept $a_1 = 1$ at $\hat{c}(\sigma) = 26$.

5 Mode normal-ordering

To obtain a more useful form of the orbifold Virasoro generators $\{L_{\hat{j}j}\}$, the next step in the orbifold program is the introduction of *mode normal-ordering*

$$\begin{aligned} & :\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)})\hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(s)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}):_M \\ & \equiv \theta((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \geq 0)\hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(s)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)})\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \\ & + \theta((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) < 0)\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)})\hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(s)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}) \quad (5.1) \end{aligned}$$

to replace the operator-product normal-ordering in Eq. (4.2).

The reordering is somewhat intricate, so we will sketch the intermediate steps. From the definition (5.1) of mode normal-ordering and the commutator (4.4c) of two twisted currents, we obtain first the relation between the product of two modes and the mode normal-ordered

product of the modes as follows:

$$\begin{aligned}
& \hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}) \\
&= : \hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \hat{J}_{n(s)\nu\hat{\ell}\ell}(n + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)}) :_M \\
&+ \theta((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) \delta_{j\ell} f_j(\sigma) (m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \\
&\quad \cdot \delta_{n(r)+n(s), 0 \bmod \rho(\sigma)} \delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)}+\frac{\hat{j}+\hat{\ell}}{f_j(\sigma)}, 0} \mathcal{G}_{n(r)\mu; -n(r), \nu}(\sigma). \quad (5.2)
\end{aligned}$$

Then using the mode expansions and the $\hat{J}\hat{J}$ operator product in Eq. (3.4c), one straightforwardly obtains the following exact relation between the two types of normal-ordering of two local currents:

$$\begin{aligned}
& : \hat{J}_{n(r)\mu\hat{j}j}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\ell}(\omega, \sigma) : - : \hat{J}_{n(r)\mu\hat{j}j}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\ell}(\omega, \sigma) :_M \\
&= f_j(\sigma) \delta_{j\ell} \mathcal{G}_{n(r)\mu; -n(r), \nu}(\sigma) \delta_{n(r)+n(s), 0 \bmod \rho(\sigma)} \delta_{\hat{j}+\hat{\ell}, 0 \bmod f_j(\sigma)} \\
&\quad \cdot \left[\frac{1}{z\omega} \left(\frac{\omega}{z} \right)^{X_{\hat{j}}} \left\{ X_{\hat{j}} \frac{z}{z-\omega} (\theta(0 \leq X_{\hat{j}} < 1) + \frac{z}{\omega} \theta(1 \leq X_{\hat{j}} < 2)) \right. \right. \\
&\quad \left. \left. + \frac{z\omega}{(z-\omega)^2} \theta(0 \leq X_{\hat{j}} < 1) - \frac{z(z-2\omega)}{\omega^2} \theta(1 \leq X_{\hat{j}} < 2) \right\} \right. \\
&\quad \left. - \frac{1}{(z-\omega)^3} \right] \quad (5.3a)
\end{aligned}$$

$$X_{\hat{j}} \equiv \frac{\bar{n}(r)}{\rho(\sigma)} + \frac{\bar{\hat{j}}}{f_j(\sigma)}, \quad 0 \leq X_{\hat{j}} < 2, \quad \forall \bar{n}(r), \bar{\hat{j}}. \quad (5.3b)$$

Here θ is the Heaviside function, and we have introduced the notational simplification $X_{\hat{j}} \equiv X_{\hat{j}n(r)}$. As $z \rightarrow \omega$, Eq. (5.3) gives the exact relation between the two types of local normal-ordered current bilinears

$$\begin{aligned}
& : \hat{J}_{n(r)\mu\hat{j}j}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\ell}(z, \sigma) : = : \hat{J}_{n(r)\mu\hat{j}j}(z, \sigma) \hat{J}_{n(s)\nu\hat{\ell}\ell}(z, \sigma) :_M \\
&+ \delta_{j\ell} f_j(\sigma) \delta_{\hat{j}+\hat{\ell}, 0 \bmod f_j(\sigma)} \mathcal{G}_{n(r)\mu; -n(r), \nu}(\sigma) \delta_{n(r)+n(s), 0 \bmod \rho(\sigma)} \\
&\quad \cdot \frac{1}{z^2} |1 - X_{\hat{j}}| (1 - |1 - X_{\hat{j}}|) \quad (5.4)
\end{aligned}$$

which is now in the desired form for application to the orbifold Virasoro generators.

With the relation (5.4), we can convert the operator-product normal-ordered forms (3.1) or (4.2) of the extended stress tensors and orbifold Virasoro generators to the following

mode-ordered forms

$$\hat{\theta}_{jj}(z, \sigma) = \frac{1}{2f_j(\sigma)} \sum_{n(r)\mu\nu} \mathcal{G}^{n(r)\mu; -n(r),\nu}(\sigma) \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} : \hat{J}_{n(r)\mu\hat{\ell}j}(z, \sigma) \hat{J}_{-n(r),\nu,\hat{j}-\hat{\ell},j}(z, \sigma) :_M + \frac{\delta_{j,0 \bmod f_j(\sigma)}}{z^2} \hat{\Delta}_{0j}(\sigma) \quad (5.5a)$$

$$\begin{aligned} \hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}) &= \frac{1}{2f_j(\sigma)} \sum_{n(r)\mu\nu} \mathcal{G}^{n(r)\mu; -n(r),\nu}(\sigma) \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} \times \\ &\times : \hat{J}_{n(r)\mu\hat{\ell}j}(p + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_j(\sigma)}) \hat{J}_{-n(r)\nu,\hat{j}-\hat{\ell},j}(m - p - \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}-\hat{\ell}}{f_j(\sigma)}) :_M \\ &+ \delta_{m+\frac{\hat{j}}{f_j(\sigma)},0} \hat{\Delta}_{0j}(\sigma) \end{aligned} \quad (5.5b)$$

where the shifts $\{\hat{\Delta}_{0j}(\sigma)\}$, given below, will be called the *conformal weights of cycle j in sector σ* . (These conformal weights were called the partial conformal weights in the early examples of Ref. [15].)

We give the explicit forms of the conformal weights $\{\hat{\Delta}_{0j}(\sigma)\}$ in a number of steps, which show various properties of these quantities:

$$\begin{aligned} \hat{\Delta}_{0j}(\sigma) &= \frac{1}{4} \sum_{n(r)\mu n(s)\nu} \mathcal{G}^{n(r)\mu; n(s)\nu}(\sigma) \mathcal{G}_{n(r)\mu; n(s)\nu}(\sigma) \sum_{\hat{j}=0}^{f_j(\sigma)-1} \times \\ &\times \{ \theta(0 \leq X_{\hat{j}} < 1) X_{\hat{j}}(1 - X_{\hat{j}}) + \theta(1 \leq X_{\hat{j}} < 2) (X_{\hat{j}} - 1)(2 - X_{\hat{j}}) \} \end{aligned} \quad (5.6a)$$

$$= \frac{1}{4} \sum_{n(r)\mu} \sum_{\hat{j}=0}^{f_j(\sigma)-1} \{ \theta(0 \leq X_{\hat{j}} < 1) X_{\hat{j}}(1 - X_{\hat{j}}) + \theta(1 \leq X_{\hat{j}} < 2) (X_{\hat{j}} - 1)(2 - X_{\hat{j}}) \} \quad (5.6b)$$

$$= \frac{1}{2} \sum_r \dim[n(r)] \sum_{\hat{j}=0}^{f_j(\sigma)-1} (1 - \frac{\bar{n}(r)}{\rho(\sigma)} - \frac{\bar{j}}{f_j(\sigma)}) \{ \frac{1}{2} (\frac{\bar{n}}{\rho(\sigma)} + \frac{\bar{j}}{f_j(\sigma)}) - \theta((\frac{\bar{n}}{\rho(\sigma)} + \frac{\bar{j}}{f_j(\sigma)}) \geq 1) \}. \quad (5.6c)$$

To obtain the second form (5.6b), we have used Eq. (2.8c) to do the sum on $n(s)\nu$, and the final step (5.6c) uses the degeneracy sum $\sum_{\mu} = \dim[\bar{n}(r)]$ in Eq. (2.4b). The bars on the quantities \hat{j} can be ignored here because the fundamental range is explicit in the summations. The second form in particular shows that the conformal weight of cycle j is nonnegative

$$\hat{\Delta}_{0j}(\sigma) \geq 0 \quad (5.7)$$

and we shall find stronger lower bounds below.

Other useful forms of the conformal weights of cycle j include:

$$\hat{\Delta}_{0j}(\sigma) = \frac{13}{12} \left(f_j(\sigma) - \frac{1}{f_j(\sigma)} \right) + \frac{1}{f_j(\sigma)} \hat{\delta}_{0j}(\sigma), \quad (5.8a)$$

$$\hat{\delta}_{0j}(\sigma) \equiv \frac{f_j(\sigma)}{2} \sum_r \dim[\bar{n}(r)] \hat{A}[\frac{\bar{n}(r)}{\rho(\sigma)}], \quad (5.8b)$$

$$\sum_r \dim[\bar{n}(r)] = 26, \quad (5.8c)$$

$$\begin{aligned} \hat{A}[\frac{\bar{n}(r)}{\rho(\sigma)}] &\equiv \left(\frac{\bar{n}(r)}{\rho(\sigma)} - \frac{1}{f_j(\sigma)} \right) \left(\theta\left(\frac{\bar{n}(r)}{\rho(\sigma)} \geq \frac{1}{f_j(\sigma)}\right) - \frac{f_j(\sigma)}{2} \frac{\bar{n}(r)}{\rho(\sigma)} \right) \\ &\quad + \sum_{\hat{j}=2}^{f_j(\sigma)-1} \left(\frac{\bar{n}(r)}{\rho(\sigma)} - \frac{\hat{j}}{f_j(\sigma)} \right) \theta\left(\frac{\bar{n}(r)}{\rho(\sigma)} \geq \frac{\hat{j}}{f_j(\sigma)}\right), \end{aligned} \quad (5.8d)$$

$$= \frac{f_j(\sigma)}{2} \sum_{\hat{j}=0}^{f_j(\sigma)-1} \left(\frac{\bar{n}(r)}{\rho(\sigma)} - \frac{\hat{j}}{f_j(\sigma)} \right) \left(\frac{\hat{j}+1}{f_j(\sigma)} - \frac{\bar{n}(r)}{\rho(\sigma)} \right) \theta\left(\frac{\hat{j}}{f_j(\sigma)} \leq \frac{\bar{n}(r)}{\rho(\sigma)} < \frac{\hat{j}+1}{f_j(\sigma)}\right). \quad (5.8e)$$

In what follows, we will refer to the quantity $\hat{\delta}_{0j}(\sigma)$ as the *conformal-weight shift* of cycle j in sector σ . The first form of the function \hat{A} in Eq. (5.8d) follows directly from Eq. (5.6c), and is easier to evaluate explicitly for small cycle length $f_j(\sigma)$. The second form of \hat{A} in Eq. (5.8e) follows by induction from the first form, and shows that

$$\hat{A}[\frac{\bar{n}(r)}{\rho(\sigma)}] \geq 0 \quad \implies \quad \hat{\delta}_{0j}(\sigma) \geq 0 \quad \implies \quad \hat{\Delta}_{0j}(\sigma) \geq \frac{13}{12} \left(f_j(\sigma) - \frac{1}{f_j(\sigma)} \right). \quad (5.9)$$

Moreover \hat{A} is a continuous function, periodic in the H'_{26} -fraction (\bar{n}/ρ) with period $(1/f_j(\sigma))$. The only zeroes of \hat{A} are at the values

$$f_j(\sigma) \frac{\bar{n}(r)}{\rho(\sigma)} \in \mathbb{Z}_{\geq 0} \quad (5.10)$$

and the maximum value in each cell is $(1/8f_j(\sigma))$.

Let us check our general result (5.8) for the previously-studied cases [3] of $H(\text{perm})_2 = \mathbb{Z}_2$ or $\mathbb{Z}_s(\text{w.s.})$, i.e. for the generalized \mathbb{Z}_2 -permutation orbifolds or the twisted open-string sectors of the orientation orbifolds. Choosing for either case the single nontrivial element with a single cycle $j = 0$ of length $f_0(\sigma) = 2$, we can easily² do the sum explicitly over $\bar{j} = 0, 1$ to obtain

$$\hat{\Delta}_{00}(\sigma) = \frac{13}{8} + \sum_r \dim[\bar{n}(r)] \left(\frac{\bar{n}(r)}{\rho(\sigma)} - \frac{1}{2} \right) \left(\left(\theta\left(\frac{\bar{n}(r)}{\rho(\sigma)} \geq \frac{1}{2}\right) - \frac{\bar{n}(r)}{\rho(\sigma)} \right) \geq \frac{13}{8} \right). \quad (5.11)$$

This is in agreement with the result in Eq. (2.3d) of Ref. [3].

²On the other hand, we find a repeated typo in Eqs. (3.36c) and (3.38b) of the earlier Ref. [18]: The terms $(n(r)/2\rho(\sigma))^2$ in each of these equations should read simply $(n(r)/\rho(\sigma))^2$, without the 2 in the denominator.

There is one more relation between the quantities discussed here which will be useful to record

$$\hat{\Delta}_{0j}(\sigma) - \hat{a}_{f_j(\sigma)} = \frac{1}{f_j(\sigma)}(\hat{\delta}_{0j}(\sigma) - 1) \quad (5.12)$$

where $\hat{a}_{f_j(\sigma)}$ is the *intercept* of cycle j in sector σ in the extended physical-state condition (1.3a).

We close this section with a simple application of our results here to the *twist-field state* $|0\rangle_{j\sigma}$ of cycle j in sector σ

$$\hat{J}_{n(r)\mu\hat{j}j}((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) |0\rangle_{j\sigma} = 0, \quad (5.13a)$$

$$\{\hat{L}_{\hat{j}j}((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) - \hat{\Delta}_{0j}(\sigma)\delta_{m+\frac{\hat{j}}{f_j(\sigma)},0}\} |0\rangle_{j\sigma} = 0, \quad (5.13b)$$

$$(\hat{L}_\sigma(m \geq 0) - \hat{\Delta}_\sigma\delta_{m,0}) |0\rangle_\sigma = 0, \quad (5.13c)$$

$$|0\rangle_\sigma \equiv \bigotimes_j |0\rangle_{j\sigma}, \quad \hat{\Delta}_\sigma \equiv \sum_j \hat{\Delta}_{0j}(\sigma). \quad (5.13d)$$

The first line in Eq. (5.13) defines this state, while the succeeding lines then follow from the mode-normal-ordered form (5.5b) of the orbifold Virasoro generators. Although the twist-field state is closely related to the physical ground-state of each cycle (see Sec. 7), we should emphasize that the twist-field state itself is not generally a physical state.

6 First discussion of the zero modes

To analyze the spectral problems associated to the extended physical-state conditions (1.3a), we need to separate out the *zero modes* $\{\hat{J}_j(0)_\sigma\}$ of cycle j in sector σ from the doubly-twisted currents

$$\hat{J}_{n(r)\mu\hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}).$$

We remind that $\mu = \mu(n(r))$ is the degeneracy index of the spectral index $n(r)$ of each element of the 26-dimensional automorphism group H'_{26} , while $\{\hat{j}j\}$ record the cycle-basis of each element of $H(\text{perm})_K$. The zero modes are special cases of what we will call the *integer-moded sequences* $\{\hat{J}_j(m)_\sigma\}$ of cycle j in sector σ .

From the twisted current algebra (4.4c), we know that the zero modes commute with all the currents, including themselves

$$[\{\hat{J}_j(0)_\sigma\}, \hat{J}_{n(r)\mu\hat{\ell}\ell}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{\ell}}{f_\ell(\sigma)})] = 0, \quad \forall j, \ell, \sigma \quad (6.1)$$

so we may alternatively refer to the zero modes as the *momenta* of cycle j in sector σ . Then it is natural to define the number of zero modes in each cycle as the *target space-time dimension* of cycle j in sector σ :

$$\hat{D}_j(\sigma) \equiv \dim\{\hat{J}_j(0)_\sigma\}. \quad (6.2)$$

The target space-time interpretation is a topic of central importance in the new string theories. In this paper, we confine ourselves only to general properties of the space-time dimensions, across all the bosonic prototypes (1.1) of the orbifold-string theories of permutation-type. The general formulae developed here are however quite model-dependent, involving the choice of subgroup H'_{26} and the particular element $\omega(\sigma) \in H'_{26}$. Beyond the simple examples of trivial H'_{26} in Sec. 10, we will return to study the target space-times of large classes of specific models in succeeding papers of this series.

From the total mode number of the doubly-twisted currents, it is straightforward to determine that the following conditions are necessary and sufficient for integer-moded sequences and hence zero modes in cycle j of sector σ :

$$\{\hat{J}_j(0)_\sigma\} : f_j(\sigma) \frac{\bar{n}(r)}{\rho(\sigma)} \in \mathbb{Z}_{\geq 0}, \quad (6.3a)$$

$$\bar{n}(r) = \frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)} \quad \text{such that } \hat{j}' = 0, 1, \dots, f_j(\sigma) - 1, \bar{n}(r) \in \{0, 1, \dots, \rho(\sigma) - 1\}. \quad (6.3b)$$

The conditions in (6.3b) are a more detailed statement of the condition in (6.3a). The explicit solutions of these conditions depend on the cycle length $f_j(\sigma)$ in sector σ of $H(\text{perm})_K$, as well as the details of $\omega(\sigma) \in H'_{26}$ reflected in the ratio $\frac{\bar{n}(r)}{\rho(\sigma)}$. We emphasize that the condition (6.3a) is the same condition under which the function $\hat{A}[\frac{\bar{n}(r)}{\rho(\sigma)}] = 0$ (see Eq. (5.10)), so that the integer-moded sequences do not contribute to the conformal-weight shift $\hat{\delta}_{0j}(\sigma)$ of cycle j in sector σ . This phenomenon is familiar in ordinary untwisted string theory, where all sequences are integer-moded and there are no conformal-weight shifts.

Using the conditions (6.3), we can give a qualitative sketch of the integer-moded sequences and zero modes as follows. We begin by noting that there are exactly two possible types of such sequences, the first of which

$$\text{type I: } \hat{J}_{0\mu(0)0j}(m) \xrightarrow{m=0} \hat{J}_{0\mu(0)0j}(0) \quad (6.4)$$

is found if and only if $\bar{n} = 0$ occurs in the spectrum of $\omega(\sigma) \in H'_{26}$. This type occurs for example (see Sec. 10) in each cycle of every sector of the “pure” permutation orbifolds with trivial H'_{26} . The second, distinct type that can arise is the following family:

$$\text{type II: } \hat{J}_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu(\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}), f_j(\sigma)-\hat{j}', j}(m+1) \xrightarrow{m=-1} \hat{J}_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu(\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}), f_j(\sigma)-\hat{j}', j}(0), \quad (6.5a)$$

$$\hat{j}' = 1, \dots, f_j(\sigma) - 1. \quad (6.5b)$$

Because the spectral index $\bar{n}(r)$ is an integer, this family occurs only when $\rho(\sigma)$ is a multiple of $f_j(\sigma)$ or vice-versa. Examples of this type with $(f_j(\sigma) = 2, \rho(\sigma) = \text{even})$ have been discussed in Ref. [3], and we will discuss both of these types more systematically in succeeding papers.

For our discussion below, it will be convenient to introduce a formal Heaviside function for the momenta:

$$\theta\{\hat{J}_j(0)_\sigma\} \equiv \begin{cases} 1 & \text{when } \omega(\sigma) \in H'_{26} \text{ allows the zero mode in } j\sigma, \\ 0 & \text{otherwise.} \end{cases} \quad (6.6)$$

This allows us to write formal expressions for the number of target space-time dimensions and the “momentum-squared” operator of cycle j in sector σ :

$$\hat{D}_j(\sigma) = \dim\{\hat{J}_j(0)_\sigma = \sum_{\bar{n}(r)\mu\hat{j}'} \theta\{\hat{J}_j(0)_\sigma\} \quad (6.7a)$$

$$\begin{aligned} \hat{P}_j^2(\sigma) \equiv & - \sum_{\mu,\nu} \theta\{\hat{J}_j(0)_\sigma\} \{ \mathcal{G}^{0\mu;0\nu}(\sigma) \hat{J}_{0\mu 0j}(0) \hat{J}_{0\nu 0j}(0) \\ & + \sum_{\hat{j}'=1}^{f_j(\sigma)-1} \mathcal{G}_{\frac{\rho_{\hat{j}'}}{f_j}, \mu; -\frac{\rho_{\hat{j}'}}{f_j}, \nu}(\sigma) \hat{J}_{\frac{\rho_{\hat{j}'}}{f_j}, \mu, f_j - \hat{j}', j}(0) \hat{J}_{-\frac{\rho_{\hat{j}'}}{f_j}, \nu, \hat{j}' - f_j, j}(0) \}. \end{aligned} \quad (6.7b)$$

For brevity, we have omitted here the sector label σ in both $f_j(\sigma)$ and $\rho(\sigma)$. The momentum-squared operator in Eq. (6.7b) will play a central role in the following analysis of the extended physical-state conditions.

7 The extended physical-state conditions

We begin this section by recalling the extended physical-state conditions for the open-string analogues of the generalized permutation orbifolds

$$\left[\frac{\mathrm{U}(1)^{26K}}{H_+} \right]_{\text{open}}, \quad H_+ \subset H(\text{perm})_K \times H'_{26}, \quad (7.1)$$

which include the twisted open-string sectors of the orientation orbifolds when $K = 2$. As obtained in the BRST quantization of Ref. [6] and quoted above in Eq. (1.3), these conditions read

$$(\hat{L}_{\hat{j}j}((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) - \hat{a}_{f_j(\sigma)} \delta_{m + \frac{\hat{j}}{f_j(\sigma)}, 0}) |\chi(\sigma)\rangle_j = 0, \quad (7.2a)$$

$$\hat{c}_j(\sigma) = 26f_j(\sigma), \quad \hat{a}_{f_j(\sigma)} = \frac{13f_j^2(\sigma) - 1}{12f_j(\sigma)} \quad (7.2b)$$

$$\bar{j} = 0, 1, \dots, f_j(\sigma) - 1, \quad j = 0, 1, \dots, N(\sigma) - 1 \quad (7.2c)$$

where $\hat{c}_j(\sigma)$ and $\hat{a}_{f_j(\sigma)}$ are respectively the cycle central charge and the intercept of cycle j in sector σ . The integer $N(\sigma)$ in Eq. (7.2c) is the number of cycles in sector σ , while the orbifold Virasoro generators $\{\hat{L}_{\hat{j}j}\}$ should now be taken in the mode normal-ordered form (5.5b).

For each cycle j in every sector σ , this system can be decomposed into the extended gauge conditions

$$\hat{L}_{\hat{j}j}((m + \frac{\hat{j}}{f_j(\sigma)}) > 0) |\chi(\sigma)\rangle_j = 0 \quad (7.3)$$

and the spectral subproblem for cycle j of sector σ :

$$(\hat{L}_{0j}(0) - \hat{a}_{f_j(\sigma)}) |\chi(\sigma)\rangle_j = 0, \quad (7.4a)$$

$$\hat{L}_{0j}(0) = \frac{1}{2f_j(\sigma)}(-\hat{P}_j^2(\sigma) + \hat{R}_j(\sigma)) + \hat{\Delta}_{0j}(\sigma), \quad (7.4b)$$

$$\begin{aligned} \hat{R}_j(\sigma) \equiv & \sum'_{n(r)\mu\nu} \mathcal{G}^{n(r)\mu; -n(r)\nu}(\sigma) \sum_{\hat{j}=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} \times \\ & \times : \hat{J}_{n(r)\mu\hat{j}j}(p + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) \hat{J}_{-n(r),\nu,-\hat{j},j}(-p - \frac{n(r)}{\rho(\sigma)} - \frac{\hat{j}}{f_j(\sigma)}) :_M. \end{aligned} \quad (7.4c)$$

The momentum-squared operator (6.7b) appears now among the terms of Eq. (7.4b), and the primed sum in the generalized number operator $\hat{R}_j(\sigma)$ denotes omission of the zero modes.

So long as the cycle-momenta $\{\hat{J}_j(0)_\sigma\}$ are not an empty set, the *physical ground-state* of cycle j in sector σ is the $\{\hat{J}_j(0)_\sigma\}$ -boosted twist-field state $|0, \hat{J}_j(0)\rangle_\sigma$

$$\hat{J}_{n(r)\mu\hat{j}j}((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) > 0) |0, \hat{J}_j(0)\rangle_\sigma = 0, \quad (7.5a)$$

$$\hat{R}_j(\sigma) |0, \hat{J}_j(0)\rangle_\sigma = \hat{L}_{jj}((m + \frac{\hat{j}}{f_j(\sigma)}) > 0) |0, \hat{J}_j(0)\rangle_\sigma = 0, \quad (7.5b)$$

$$\hat{P}_j^2(\sigma) |0, \hat{J}_j(0)\rangle_\sigma = \hat{P}_j^2(\sigma)_{(0)} |0, \hat{J}_j(0)\rangle_\sigma, \quad (7.5c)$$

$$\hat{P}_j^2(\sigma)_{(0)} = 2(\hat{\delta}_{0j}(\sigma) - 1) \geq -2 \quad (7.5d)$$

where the *ground-state momentum-squared* of cycle j in sector σ is given in Eq. (7.5d). To obtain this result, we used Eqs. (7.4a,b), and the relation (5.12) between the fundamental constants of the cycle. The explicit form of the conformal-weight shift $\hat{\delta}_{0j}(\sigma) \geq 0$ is given in Eq. (5.8b), and we see from Eq. (7.5d) that the conformal-weight shift does indeed measure a shift from the ground-state momentum-squared $P^2 = -2$ of an ordinary untwisted open string.

According to Eqs. (4.4b) and (7.4b), the excited states of cycle j in sector σ exhibit the *level-spacing*

$$\Delta(\hat{P}_j^2(\sigma)) = \Delta(\hat{R}_j(\sigma)) = 2f_j(\sigma) |m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}|. \quad (7.6)$$

These are the increments of mass-squared associated to the addition of any negatively-moded current

$$\hat{J}_{n(r)\mu\hat{j}j}((m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}) < 0)$$

to products of other such currents on the ground-state of cycle j in sector σ .

We turn next to the extended physical-state conditions of all the closed-string sectors of the generalized permutation orbifolds

$$\frac{\text{U}(1)^{26K}}{H_+}, \quad H_+ \subset H(\text{perm})_K \times H'_{26} \quad (7.7)$$

whose cycles also live at cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$ and sector central charge $\hat{c}(\sigma) = 26K$. In these cases we have a left- and right-mover copy of the extended physical-state conditions (7.2), which we write as

$$\begin{aligned}\hat{L}_{jj}^L((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) |\chi(\sigma)\rangle_j &= \hat{L}_{jj}^R((m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0) |\chi(\sigma)\rangle_j \\ &= \hat{a}_{f_j(\sigma)} \delta_{m + \frac{\hat{j}}{f_j(\sigma)}, 0} |\chi(\sigma)\rangle_j,\end{aligned}\tag{7.8a}$$

$$\hat{a}_{f_j(\sigma)} = \frac{13f_j^2(\sigma) - 1}{12f_j(\sigma)},\tag{7.8b}$$

$$\hat{L}_{0j}^L(0) = \frac{1}{2f_j(\sigma)} (\hat{P}_j^2(\sigma)^L + \hat{R}_j(\sigma)^L + \hat{\Delta}_{0j}(\sigma)),\tag{7.8c}$$

$$\hat{L}_{0j}^R(0) = \frac{1}{2f_j(\sigma)} (\hat{P}_j^2(\sigma)^R + \hat{R}_j(\sigma)^R + \hat{\Delta}_{0j}(\sigma)).\tag{7.8d}$$

The extended Virasoro generators $\{\hat{L}^L\}$ and $\{\hat{L}^R\}$ involve the twisted left- and right-mover currents $\{\hat{J}^L\}$ and $\{\hat{J}^R\}$ respectively.

Following Ref. [3], we study only the case of decompactified zero modes, with the ordinary left-right identifications:

$$\hat{J}_j^R(0)_\sigma = \hat{J}_j^L(0)_\sigma = \frac{1}{\sqrt{2}} \hat{J}_j(0)_\sigma,\tag{7.9a}$$

$$\hat{P}_j^2(\sigma)^R = \hat{P}_j^2(\sigma)^L = \frac{1}{2} \hat{P}_j^2(\sigma).\tag{7.9b}$$

The closed-string momenta $\{\hat{J}_j(0)_\sigma\}$ and momentum-squared $\hat{P}_j^2(\sigma)$ appear on the right side of Eqs. (7.9 a,b), and $\hat{P}_j^2(\sigma)$ has exactly the form (6.7b) when expressed in terms of $\{\hat{J}_j(0)_\sigma\}$. Then the extended physical-state conditions (7.8) can be put in the form

$$\hat{P}_j^2(\sigma) |\chi(\sigma)\rangle_j = 2(\hat{P}_j^2(\sigma)_{(0)} + \hat{R}_j^L(\sigma)) |\chi(\sigma)\rangle_j,\tag{7.10a}$$

$$(\hat{R}_j^R(\sigma) - \hat{R}_j^L(\sigma)) |\chi(\sigma)\rangle_j = 0,\tag{7.10b}$$

$$\hat{L}_{jj}^R((m + \frac{\hat{j}}{f_j(\sigma)}) > 0) |\chi(\sigma)\rangle_j = \hat{L}_{jj}^L((m + \frac{\hat{j}}{f_j(\sigma)}) > 0) |\chi(\sigma)\rangle_j = 0\tag{7.10c}$$

where $\hat{P}_j^2(\sigma)_{(0)}$ is defined in Eq. (7.5d), and Eq. (7.10b) is the level-matching condition for cycle j in sector σ . Assuming again that the momenta are not an empty set, we find that the physical closed-string ground-state $|0, \hat{J}_j(0)\rangle_\sigma$ of cycle j has ground-state momentum-squared

$$\hat{P}_j^2(\sigma)^{\text{closed}}(0) = 2\hat{P}_j^2(\sigma)_{(0)} = 4(\hat{\delta}_{0j}(\sigma) - 1) \geq -4\tag{7.11}$$

that is, twice the ground-state momentum-squared of the corresponding twisted open string.

In what follows we will explicitly discuss only the twisted open-string cases, but the corresponding closed-string cases can easily be obtained from the results above.

8 The reduced formulation at $c_j(\sigma) = 26$

Recall that the $\hat{c}(\sigma) = 52$ physical spectral problems have an equivalent, reduced description [3,4] at reduced central charge $c(\sigma) = 26$. In this and the following section, we generalize this result to include all the bosonic prototypes (1.1) at sector central charge $\hat{c}(\sigma) = 26K$ and cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$. In particular, we find the equivalent, reduced formulation of each cycle j at *reduced cycle central charge* $c_j(\sigma) = 26$, independent of the cycle. We emphasize with the earlier references that *this equivalence holds only for the cycle dynamics of the orbifold-string theories as described by the extended physical-state conditions*, and *not* for the underlying orbifold CFT's themselves. One advantage of the original description at $\hat{c}(\sigma) = 26K$ is locality [1], which provides twisted local vertex operators [13,15,18,21,4,5]. With Ref. [3] we shall see however that the zero modes and target space-time dimensions are *invariant* under the reduction, and we shall emphasize in succeeding papers that the target space-time properties of the theories are more easily studied in the reduced description.

In this paper we organize the discussion of the equivalent, reduced formulation into two parts. In the present section we work out the operators of the reduced formulation at $c_j(\sigma) = 26$, leaving for Sec. 9 the reduced physical-state conditions and the equivalence of the two formulations at the string level.

The reduced (unhatted) operators of cycle j in sector σ are defined by the following map

$$L_j(M_j) \equiv f_j(\sigma) \hat{L}_{\hat{j}j}(m + \frac{\hat{j}}{f_j(\sigma)}) - \frac{13}{12}(f_j(\sigma)^2 - 1) \delta_{m + \frac{\hat{j}}{f_j(\sigma)}, 0}, \quad (8.1a)$$

$$J_{n(r)\mu j}(M_j + f_j(\sigma) \frac{n(r)}{\rho(\sigma)}) \equiv \hat{J}_{n(r)\mu \hat{j}j}(m + \frac{n(r)}{\rho(\sigma)} + \frac{\hat{j}}{f_j(\sigma)}), \quad (8.1b)$$

$$M_j \equiv f_j(\sigma)m + \bar{\hat{j}} \in \mathbb{Z} \quad (8.1c)$$

in terms of the hatted operators above. Since $m \in \mathbb{Z}$ and $\bar{\hat{j}} \in 0, 1, \dots, f_j(\sigma) - 1$, the capitalized quantities M_j cover the integers once for each cycle j , and indeed the map is one-to-one at each fixed (j, σ) . This map is in fact a modest generalization of the (inverse of) the order- λ orbifold-induction procedure of Borisov, Halpern, and Schweigert [7].

Then we find from Eq. (4.4) the explicit algebra of the reduced operators:³

$$[L_j(M), L_\ell(N)] = \delta_{j\ell} \{ (M - N)L_j(M + N) + \frac{26}{12}M(M^2 - 1)\delta_{M+N,0} \}, \quad (8.2a)$$

$$[L_j(M), J_{n(r)\mu\ell}(N + f_\ell(\sigma)\frac{n(r)}{\rho(\sigma)})] = -\delta_{j\ell}(N + f_\ell(\sigma)\frac{n(r)}{\rho(\sigma)})J_{n(r)\mu\ell}(M + N + f_\ell(\sigma)\frac{n(r)}{\rho(\sigma)}), \quad (8.2b)$$

$$\begin{aligned} & [J_{n(r)\mu j}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}), J_{n(s)\nu\ell}(N + f_\ell(\sigma)\frac{n(s)}{\rho(\sigma)})] \\ &= \delta_{j\ell}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)})\delta_{n(r)+n(s), 0 \bmod \rho(\sigma)}\delta_{M+N+f_j(\sigma)\frac{n(r)+n(s)}{\rho(\sigma)}, 0}\mathcal{G}_{n(r)\mu; -n(r)\nu}(\sigma), \end{aligned} \quad (8.2c)$$

$$J_{n(r)\pm\rho(\sigma), \mu j}(m + f_j(\sigma)\frac{n(r)\pm\rho(\sigma)}{\rho(\sigma)}) = J_{n(r)\mu}(m \pm f_j(\sigma) + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}). \quad (8.2d)$$

In particular, Eq. (8.2a) shows that the reduced generators $\{L_j(M)\}$ satisfy an ordinary Virasoro algebra with reduced cycle central charge $c_j(\sigma) = 26$ for each cycle j in any sector σ . The total reduced Virasoro generators of sector σ

$$L_\sigma(M) \equiv \sum_j L_j(M) \quad (8.3)$$

are then also Virasoro with reduced sector central charge

$$c(\sigma) = \sum_j c_j(\sigma) = 26N(\sigma) \quad (8.4)$$

where $N(\sigma)$ is the number of cycles in sector σ .

With Eq. (5.5b), the map also gives the explicit form of the reduced Virasoro generators of each cycle at $c_j(\sigma) = 26$ in terms of the reduced currents:

$$\begin{aligned} L_j(M) &= \delta_{M,0}\hat{\delta}_{0j}(\sigma) + \frac{1}{2} \sum_{n(r)\mu\nu} \mathcal{G}^{n(r)\mu; -n(r)\nu}(\sigma) \sum_{p \in \mathbb{Z}} \times \\ &\quad \times :J_{n(r)\mu j}(P + f_j(\sigma)\frac{n(r)}{\rho(\sigma)})J_{-n(r), \nu j}(M - P - f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) :_M, \end{aligned} \quad (8.5a)$$

$$\begin{aligned} \hat{\delta}_{0j}(\sigma) &= \frac{1}{2} \sum_r \dim[\bar{n}(r)] \left\{ (f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} - 1)(\theta(f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} \geq 1) - f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)}) \right. \\ &\quad \left. + \sum_{\hat{j}=2}^{f_j(\sigma)-1} (f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} - \hat{j})\theta(f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} \geq \hat{j}) \right\}, \end{aligned} \quad (8.5b)$$

$$= \frac{1}{4} \sum_r \dim[\bar{n}(r)] \sum_{\hat{j}=0}^{f_j(\sigma)-1} (f_j(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)} - \hat{j})(\hat{j} + 1 - \frac{f_j(\sigma)\bar{n}(r)}{\rho(\sigma)})\theta(\hat{j} \leq \frac{f_j(\sigma)\bar{n}(r)}{\rho(\sigma)} < \hat{j} + 1), \quad (8.5c)$$

$$j = 0, 1, \dots, N(\sigma) - 1, \quad \sum_j f_j(\sigma) = K, \quad \sum_r \dim[\bar{n}(r)] = 26. \quad (8.5d)$$

³ In Eq. (4.2e) of Ref. [3], there is a missing factor $(M + 2\frac{n(r)}{\rho(\sigma)})$, which is now included in (8.2c) when $f_j(\sigma) = 2$.

Here the expressions for the conformal-weight shifts $\hat{\delta}_{0j}(\sigma)$ are the same as those given in Eq. (5.8), now slightly rearranged to emphasize the scaling into the characteristic ratio $(f_j(\sigma)\frac{n(r)}{\rho(\sigma)})$ of the reduced formulation. The explicit form of the mode-normal ordering here

$$\begin{aligned} & :J_{n(r)\mu j}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)})J_{n(s)\nu\ell}(N + f_\ell(\sigma)\frac{n(s)}{\rho(\sigma)}) :_M \\ & = \theta((M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) \geq 0)J_{n(s)\nu\ell}(N + f_\ell(\sigma)\frac{n(s)}{\rho(\sigma)})J_{n(r)\mu j}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) \\ & \quad + \theta((M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) < 0)J_{n(r)\mu j}(M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)})J_{n(s)\nu\ell}(N + f_\ell(\sigma)\frac{n(s)}{\rho(\sigma)}) \quad (8.6) \end{aligned}$$

is also obtained as the image of the original mode-ordering in Eq. (5.2). This result reflects the simple fact that the map (8.1) preserves the sign of the mode number of each operator.

As emphasized for the case of a single cycle of length two in Ref. [3], the reduced Virasoro generators (8.5) at $c_j(\sigma) = 26$ are generically unconventional in form: We remind that the spectral data of each element $\omega(\sigma) \in H'_{26}$ is recorded in the conventional orbifold fraction $n(r)/\rho(\sigma)$. The cycle length $f_j(\sigma)$ in the characteristic ratio $(f_j(\sigma)\frac{n(r)}{\rho(\sigma)})$ seen here represents the effect on $\omega(\sigma)$ due to the unwinding of cycle j in each element of the basic permutation group $H(\text{perm})_K$ of the orbifold-string theories. Beyond the orbifold program and the reduction procedure described here, we are presently unaware of any alternate path to these new Virasoro generators.

Two further remarks are relevant before discussing the reduced form of the extended physical-state condition in the following section. The first remark concerns the *target space-time structure* of these theories, which is *invariant* under the reduction. In particular, each zero mode and hence the space-time dimension of each cycle is unchanged by the map

$$J_j(0)_\sigma = \hat{J}_j(0)_\sigma, \quad \theta\{J_j(0)_\sigma\} = \theta\{\hat{J}_j(0)_\sigma\}, \quad (8.7a)$$

$$D_j(\sigma) = \hat{D}_j(\sigma) \quad (8.7b)$$

and in fact the reduced formulation gives us a slightly more uniform labeling of the momenta and the momentum-squared operator of each cycle:

$$\{J_j(0)_\sigma\} : \quad J_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu(\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)})j}(0), \quad \hat{j}' = 0, 1, \dots, f_j(\sigma) - 1, \quad (8.8a)$$

$$\begin{aligned} P_j^2(\sigma) &= \hat{P}_j^2(\sigma) \\ &= - \sum_{\mu, \nu} \sum_{\hat{j}'=0}^{f_j(\sigma)-1} \theta\{J_j(0)_\sigma\} \mathcal{G}_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu; \frac{-\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \nu}(\sigma) J_{\frac{\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \mu j}(0) J_{\frac{-\rho(\sigma)\hat{j}'}{f_j(\sigma)}, \nu j}(0). \quad (8.8b) \end{aligned}$$

Here the type I zero modes are included at $\hat{j}' = 0$ (see Eq. (6.5b)), and the form of the momentum-squared operator $\hat{P}_j^2(\sigma)$ at $\hat{c}(\sigma) = 26f_j(\sigma)$ was given in Eq. (6.7b). Similarly,

the reduced number operator $R_j(\sigma)$ in the decomposition of $L_j(0)$ is the same

$$L_j(0) = \frac{1}{2}(-P_j^2(\sigma) + R_j(\sigma)) + \hat{\delta}_{0j}(\sigma), \quad (8.9a)$$

$$\begin{aligned} R_j(\sigma) &= \hat{R}_j(\sigma) \\ &= \left(\sum_{n(r)\mu\nu} \sum_{P \in \mathbb{Z}} \right)' : J_{n(r)\mu j} (P + f_j(\sigma) \frac{n(r)}{\rho(\sigma)}) J_{-n(r), \nu j} (-P - f_j(\sigma) \frac{n(r)}{\rho(\sigma)}) :_M \end{aligned} \quad (8.9b)$$

where the $\hat{c}(\sigma) = 26f_j(\sigma)$ form of $\hat{R}_j(\sigma)$ was given in Eq. (7.4c).

The second remark concerns the twist-field state $|0\rangle_{j\sigma}$ whose definition in Eq. (5.13) translates in the $c_j(\sigma) = 26$ formulation to

$$J_{n(r)\mu j}((M + f_j(\sigma) \frac{n(r)}{\rho(\sigma)}) \geq 0) |0\rangle_{j\sigma} = 0. \quad (8.10)$$

Under the action of the reduced Virasoro generators, we find then that the conformal weights of this state are shifted as follows

$$(L_j(M \geq 0) - \delta_{M,0} \hat{\delta}_{0j}(\sigma)) |0\rangle_{j\sigma} = 0, \quad (8.11a)$$

$$(L_\sigma(M \geq 0) - \delta_{M,0} \hat{\delta}_\sigma) |0\rangle_\sigma = 0, \quad (8.11b)$$

$$|0\rangle_\sigma = \bigotimes_j |0\rangle_{j\sigma}, \quad \hat{\delta}_\sigma \equiv \sum_j \hat{\delta}_{0j}(\sigma) \quad (8.11c)$$

where $L_\sigma(M) = \sum_j L_j(M)$ are the total Virasoro generators of sector σ .

In summary so far, the conformal-field-theoretic shifts we have observed in the reduction

$$\hat{c}_j(\sigma) = 26f_j(\sigma) \longrightarrow c_j(\sigma) = 26, \quad (8.12a)$$

$$\hat{\Delta}_{0j}(\sigma) \longrightarrow \hat{\delta}_{0j}(\sigma), \quad (8.12b)$$

$$j = 0, 1, \dots, N(\sigma) - 1, \quad (8.12c)$$

$$\hat{c}(\sigma) = 26K \longrightarrow c(\sigma) = 26N(\sigma) \quad (8.12d)$$

are generalizations of the (inverse of) the central charge and conformal-weight shifts found in the original orbifold-induction procedure [7] and Ref. [3].

9 Equivalent $c_j(\sigma) = 26$ description of the physical states

There are two interpretations of the map (8.1) and its inverse. In the original conformal-field-theoretic interpretation of Ref. [7], the results of the previous section provide the construction (by relabeling) of one *distinct* CFT in terms of another. We are not directly concerned with this CFT interpretation here. There is however a second interpretation of the reduction

procedure for the orbifold-*string* theories of permutation-type, as restricted by the extended physical-state conditions (7.2) or (7.8) at $\hat{c}_j(\sigma) = 26f_j(\sigma)$. In this string-theoretic interpretation, the map gives us a *completely equivalent* $c_j(\sigma) = 26$ description of the physical spectrum of each cycle j in every sector σ of the new string theories.

Indeed, it is easily checked that all the components $\tilde{j} = 0, 1, \dots, f_j(\sigma) - 1$ of the extended physical-state conditions (7.2) of cycle j map directly onto the simpler physical-state condition of cycle j in the reduced $c_j(\sigma) = 26$ description:

$$(L_j(M \geq 0) - \delta_{M,0}) |\chi(\sigma)\rangle_j = 0, \quad (9.1a)$$

$$j = 0, 1, \dots, N(\sigma) - 1. \quad (9.1b)$$

Here the reduced mode-ordered Virasoro generators $\{L_j(M)\}$ of cycle j are given in Eq. (8.5) and $N(\sigma)$ is the number of cycles in sector σ . Note that the reduced physical state conditions (9.1) are *conventional*, in that they exhibit *unit intercept* for each cycle-string j . We finally emphasize with Ref. [3] that the states described here are *exactly the same physical states* $|\chi(\sigma)\rangle_j$ – now rewritten in terms of the reduced currents – which were originally defined by the extended physical-state conditions (7.2) of the unreduced formulation at $\hat{c}_j(\sigma) = 26f_j(\sigma)$.

For example, assuming again that the zero-modes $\{J_j(0)_\sigma\} = \{\hat{J}_j(0)_\sigma\}$ of cycle j in sector σ are not an empty set, the physical ground-state of cycle j in sector σ is the *same* momentum-boosted twist-field state as determined earlier (see Eq. (7.5)) in the unreduced formulation:

$$|0, J_j(0)\rangle_\sigma = |0, \hat{J}_j(0)\rangle_\sigma, \quad (9.2a)$$

$$J_{n(r)\mu j}((M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)} > 0) |0, J_j(0)\rangle_\sigma = 0, \quad (9.2b)$$

$$L_j(M > 0) |0, J_j(0)\rangle_\sigma = 0, \quad (9.2c)$$

$$P_j^2(\sigma) |0, J_j(0)\rangle_\sigma = P_j^2(\sigma)_{(0)} |0, J_j(0)\rangle_\sigma, \quad (9.2d)$$

$$P_j^2(\sigma)_{(0)} = \hat{P}_j^2(\sigma)_{(0)} = 2(\hat{\delta}_{0j}(\sigma) - 1) \geq -2. \quad (9.2e)$$

Similarly, using the commutator (8.2b), the decomposition (8.9a) and the reduced physical-state conditions (9.1), one finds the level-spacing in the reduced description of cycle j as

$$\Delta(P_j^2(\sigma)) = \Delta(R_j(\sigma)) = 2|M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}|. \quad (9.3)$$

This spacing results when a negatively-moded reduced current $J_{n(r)\mu j}((M + f_j(\sigma)\frac{n(r)}{\rho(\sigma)}) < 0)$ is added to any previous state $J \dots J |0, J_j(0)\rangle_\sigma$. Recalling that $M = f_j(\sigma)m + \hat{j}$ in the fundamental range of \hat{j} , these increments are recognized as the *same* increments (7.6) obtained in the $\hat{c}_j(\sigma) = 26f_j(\sigma)$ description of the cycle.

Finally a right-mover copy of the reduced physical-state conditions at $c_j(\sigma) = 26$ must be added

$$(L_j^L(M \geq 0) - \delta_{M,0}) |\chi(\sigma)\rangle_j = (L_j^R(M \geq 0) - \delta_{M,0}) |\chi(\sigma)\rangle_j = 0 \quad (9.4)$$

to obtain the reduced description of the physical states of the closed-string sectors of the generalized permutation-orbifolds. Again, these are the *same* physical states described at $\hat{c}_j(\sigma) = 26f_j(\sigma)$ in Eq. (7.8). The reduced system (9.4) decomposes as follows

$$J_j^R(0)_\sigma = J_j^L(0)_\sigma = \frac{1}{\sqrt{2}}J_j(0), \quad (9.5a)$$

$$P_j^2(\sigma)^R = P_j^2(\sigma)^L = \frac{1}{2}P_j^2(\sigma), \quad (9.5b)$$

$$P_j^2(\sigma) |\chi(\sigma)\rangle_j = 2(P_j^2(\sigma)_{(0)} + R_j^L(\sigma)) |\chi(\sigma)\rangle_j, \quad (9.5c)$$

$$(R_j^R(\sigma) - R_j^L(\sigma)) |\chi(\sigma)\rangle_j = 0, \quad (9.5d)$$

$$L_j^R(M > 0) |\chi(\sigma)\rangle_j = L_j^L(M > 0) |\chi(\sigma)\rangle_j = 0, \quad (9.5e)$$

$$P_j^2(\sigma)_{(0)}^{\text{closed}} = \hat{P}_j^2(\sigma)_{(0)}^{\text{closed}} = 4(\hat{\delta}_{0j}(\sigma) - 1) \geq -4 \quad (9.5f)$$

so that the *same* value (7.11) of the ground-state momentum-squared is obtained as well in the reduced formulation of the closed-string sectors.

10 Example: The “pure” permutation orbifolds

The general formulae above are model-dependent, both in regard to $H(\text{perm})_K$ (where we have been explicit), and especially on the choice of element $\omega(\sigma) \in H'_{26}$ (whose information is encoded in the quantities $\{n(r)\mu, \mathcal{G}(\sigma)\}$). These formulae will be used extensively in succeeding papers to study large classes of models with explicit, non-trivial H'_{26} .

In this paper, however, we limit ourselves only to the very simplest orbifold-string theories of permutation-type

$$\left[\frac{U(1)^{26K}}{H(\text{perm})_K} \right]_{\text{open}}, \quad \left[\frac{U(1)^{26K}}{H(\text{perm})_K} \right] \quad (10.1)$$

that is, the closed- and open-string analogues of the “pure” permutation orbifolds with trivial H'_{26} . The sectors σ of these orbifolds are described by the equivalence classes of $H(\text{perm})_K$ alone, and the open-string sectors of the orientation-orbifolds $U(1)^{26}/\mathbb{Z}_2(w.s.)$ are included in the open-string analogues when $K = 2$.

For these cases, the solutions to the H'_{26} eigenvalue problem (2.4a) is very simple:

$$\omega(\sigma) = U(\sigma) = 1, \quad \rho(\sigma) = 1, \quad \bar{n}(0) = 0, \quad (10.2a)$$

$$\mu = a = 0, 1, \dots, 25, \quad (10.2b)$$

$$\mathcal{G}(\sigma) = \mathcal{G}(\sigma) = G_{ab} = -\eta_{ab}, \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad (10.2c)$$

$$\sum_\mu = \sum_a = \dim[\bar{n}(0)] = 26. \quad (10.2d)$$

Here we have chosen the single spectral index in $\{\bar{n}(r)\}$ to be $\bar{n}(0) = 0$, with degeneracy 26 described by the degeneracy index $\mu = a$. The 26-dimensional Minkowski metric η is inherited directly from the untwisted copies of the critical closed-string $U(1)^{26}$.

With the data (10.3) for trivial H'_{26} , the orbifold Virasoro generators and the algebras of twisted sector σ are easily read from Eqs. (4.4), (5.5b) and (5.8):

$$\begin{aligned} \hat{L}_{jj}(m + \frac{j}{f_j(\sigma)}) &= \delta_{m+\frac{j}{f_j(\sigma)},0} \hat{\Delta}_{0j}(\sigma) \\ &\quad - \frac{1}{2f_j(\sigma)} \eta^{ab} \sum_{\hat{\ell}=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} : \hat{J}_{0a\hat{\ell}j}(p + \frac{\hat{\ell}}{f_j(\sigma)}) \hat{J}_{0b,\hat{j}-\hat{\ell},j}(m-p + \frac{j-\hat{\ell}}{f_j(\sigma)}) :_M, \end{aligned} \quad (10.3a)$$

$$\hat{\delta}_{0j}(\sigma) = 0, \quad \hat{\Delta}_{0j}(\sigma) = \frac{13}{12} \left(f_j(\sigma) - \frac{1}{f_j(\sigma)} \right), \quad (10.3b)$$

$$\begin{aligned} [\hat{L}_{jj}(m + \frac{j}{f_j(\sigma)}), \hat{L}_{jj}(n + \frac{\hat{\ell}}{f_\ell(\sigma)})] \\ = \delta_{j\ell} \left\{ (m-n - \frac{j-\hat{\ell}}{f_j(\sigma)}) \hat{L}_{\hat{j}+\hat{\ell},j}(m+n + \frac{j+\hat{\ell}}{f_j(\sigma)}) \right. \\ \left. + \frac{26f_j(\sigma)}{12} (m + \frac{j}{f_j(\sigma)}) ((m + \frac{j}{f_j(\sigma)})^2 - 1) \delta_{m+n+\frac{j+\hat{\ell}}{f_j(\sigma)},0} \right\}, \end{aligned} \quad (10.3c)$$

$$[\hat{L}_{jj}(m + \frac{j}{f_j(\sigma)}), \hat{J}_{0a\hat{\ell}j}(n + \frac{\hat{\ell}}{f_\ell(\sigma)})] = -\delta_{j\ell} (n + \frac{\hat{\ell}}{f_\ell(\sigma)}) \hat{J}_{0a,\hat{j}+\hat{\ell},j}(m+n + \frac{j+\hat{\ell}}{f_j(\sigma)}), \quad (10.3d)$$

$$[\hat{J}_{0a\hat{j}j}(m + \frac{j}{f_j(\sigma)}), \hat{J}_{0b\hat{\ell}j}(n + \frac{\hat{\ell}}{f_\ell(\sigma)})] = \delta_{j\ell} \eta_{ab} f_j(\sigma) (n + \frac{\hat{\ell}}{f_\ell(\sigma)}) \delta_{m+n+\frac{j+\hat{\ell}}{f_j(\sigma)},0}, \quad (10.3e)$$

$$\begin{aligned} \bar{j} &= 0, 1, \dots, f_j(\sigma) - 1, \quad a = 0, 1, \dots, 25, \\ j &= 0, 1, \dots, N(\sigma) - 1, \quad \sum_j f_j(\sigma) = K. \end{aligned} \quad (10.3f)$$

We remind that $f_j(\sigma)$ is the length of cycle j in sector σ and the summation convention is assumed for repeated indices a, b . To this list, we may add the periodicity conditions

$$\hat{L}_{\hat{j}+f_j(\sigma),j}(m + \frac{j \pm f_j(\sigma)}{f_j(\sigma)}) = \hat{L}_{jj}(m \pm 1 + \frac{j}{f_j(\sigma)}), \quad (10.4a)$$

$$\hat{J}_{0a,\hat{j} \pm f_j(\sigma),j}(m + \frac{j \pm f_j(\sigma)}{f_j(\sigma)}) = \hat{J}_{0a\hat{j}j}(m \pm 1 + \frac{j}{f_j(\sigma)}) \quad (10.4b)$$

and the adjoint operations in these theories

$$\hat{J}_{0a\hat{j}j}(m + \frac{j}{f_j(\sigma)})^\dagger = \hat{J}_{0a,-\hat{j},j}(-m - \frac{j}{f_j(\sigma)}), \quad (10.5a)$$

$$\hat{L}_{\hat{j}j}(m + \frac{j}{f_j(\sigma)})^\dagger = \hat{L}_{-\hat{j},j}(-m - \frac{j}{f_j(\sigma)}), \quad \hat{L}_{0j}(m)^\dagger = \hat{L}_{0j}(-m), \quad (10.5b)$$

$$\|\hat{J}_{0a\hat{\ell}j}((m + \frac{\hat{\ell}}{f_\ell(\sigma)}) < 0) |0, \hat{J}_l(0)\rangle_\sigma\|^2 = G_{aa} f_\ell(\sigma) |m + \frac{\hat{\ell}}{f_\ell(\sigma)}| \||0, \hat{J}_l(0)\rangle_\sigma\|^2. \quad (10.5c)$$

In fact, Ref. [11] gives a form for the adjoint operation in any current-algebraic orbifold, but we give this result here only for these simple cases. Note in particular that the only negative-norm basis states are associated here to the time-direction $a = 0$ with $G_{00} = -1$.

We also remind that the cycle and sector central charges of these theories are

$$\hat{c}_j(\sigma) = 26f_j(\sigma), \quad \hat{c}(\sigma) = \sum_j \hat{c}_j(\sigma) = 26K. \quad (10.6)$$

The system above is therefore an abelian limit of the results given for the pure WZW permutation orbifolds [15] with $\hat{c}_j(\sigma) = c_g f_j(\sigma)$ and $\hat{c}(\sigma) = K c_g$, where c_g is the central charge of the affine–Sugawara construction [22] on g .

The zero-modes (momenta) of cycle j in sector σ are entirely of type I (see Sec. 6) in all these cases

$$\{\hat{J}_j(0)_\sigma\} = \{\hat{J}_{0a0j}(0), a = 0, 1, \dots, 25\}, \quad (10.7a)$$

$$\hat{D}_j(\sigma) = 26, \quad \hat{D}(\sigma) = \sum_j \hat{D}_j(\sigma) = 26N(\sigma), \quad (10.7b)$$

$$\hat{P}_j^2(\sigma) = \eta^{ab} \hat{J}_{0a0j}(0) \hat{J}_{0b0j}(0), \quad (10.7c)$$

$$\hat{P}_j^2(\sigma)_{(0)} = -2, \quad \Delta(\hat{P}_j^2(\sigma)) = 2f_j(\sigma)|m + \frac{\hat{j}}{f_j(\sigma)}| \quad (10.7d)$$

where $\hat{D}_j(\sigma) = 26$ is the target space-time dimension of cycle j in sector σ and $N(\sigma)$ is the number of cycles in sector σ . We note in particular that each cycle j , though described here at $\hat{c}_j(\sigma) = 26f_j(\sigma)$, has exactly 26 space-time dimensions at ground-state mass-squared -2 , just as in an ordinary untwisted critical open string. Of course, the data given so far is for the open-string sectors of $[U(1)^{26K}/H(\text{perm})_K]_{\text{open}}$, whereas a right-mover copy is needed to describe the closed-string sectors of $U(1)^{26K}/H(\text{perm})_K$. In the latter cases we find that each cycle has 26 left- and right-mover momenta and $\hat{P}_j^2(\sigma)^{\text{closed}} = -4$, again the same as an ordinary untwisted critical closed string. The open- and closed-string forms of the extended physical-state conditions are given respectively in Eqs. (7.2) and (7.8).

Let us turn finally to the equivalent, reduced formulation of these theories at reduced cycle central charge $c_j(\sigma) = 26$, where one finds from Eqs. (8.2),(8.5) and (10.2) that

$$L_j(M) = -\frac{1}{2}\eta^{ab} \sum_{P \in \mathbb{Z}} :J_{0aj}(P) J_{0bj}(M-P):_M, \quad (10.8a)$$

$$[L_j(M), L_\ell(N)] = \delta_{j\ell} \left\{ (M-N)L_j(M+N) + \frac{26}{12}M(M^2-1)\delta_{M+N,0} \right\}, \quad (10.8b)$$

$$[L_j(M), J_{0a\ell}(N)] = -\delta_{j\ell} N J_{0a\ell}(M+N), \quad (10.8c)$$

$$[J_{0aj}(M), J_{0b\ell}(N)] = -\delta_{j\ell} \eta_{ab} N \delta_{M+N,0}, \quad (10.8d)$$

$$a = 0, 1, \dots, 25, \quad j = 0, 1, \dots, N(\sigma) - 1. \quad (10.8e)$$

As discussed more generally in Sec. 8, the zero-modes (momenta) in the reduced formulation are isomorphic to those in the unreduced formulation:

$$\{J_j(0)_\sigma\} = \{\hat{J}_j(0)_\sigma\} = \{J_{0aj}(0), a = 0, 1, \dots, 25\}, \quad (10.9a)$$

$$D_j(\sigma) = \hat{D}_j(\sigma) = 26 = c_j(\sigma), \quad D(\sigma) = \hat{D}(\sigma) = 26N(\sigma) = c(\sigma), \quad (10.9b)$$

$$P_j^2(\sigma) = \hat{P}_j^2(\sigma) = \eta^{ab} J_{0a0j}(0) J_{0b0j}(0), \quad (10.9c)$$

$$P_j^2(\sigma)_{(0)} = \hat{P}_j^2(\sigma)_{(0)} = -2, \quad \Delta P_j^2(\sigma)_{(0)} = \Delta \hat{P}_j^2(\sigma)_{(0)} = 2|M|. \quad (10.9d)$$

The last result is the level-spacing induced by adding an extra negatively-moded current $J_{0aj}(M < 0)$ to a lower-level state and, recalling that $M_j = mf_j(\sigma) + \tilde{j}$, one checks that this level-spacing is indeed the same as that given in Eq. (10.7d) for the unreduced currents.

We end our technical discussion with the adjoints and norms in the reduced formulation

$$J_{0aj}(M)^\dagger = J_{0aj}(-M), \quad L_j(M)^\dagger = L_j(-M), \quad (10.10a)$$

$$\|J_{0a\ell}(M < 0) |0, J_l(0)_\sigma\rangle\|^2 = G_{aa}|M| \| |0, J_l(0)_\sigma\rangle\|^2 \quad (10.10b)$$

where the adjoints are the map of Eqs. (10.5a,b) and the norms are computed from the adjoints. Again using $M_\ell = f_\ell(\sigma)m + \tilde{\ell}$, we see that the norms are the same as those computed in Eq. (10.5c) for the original formulation. Similarly, of course, the inner product of any two states are the same in the reduced and unreduced formulations.

Taken together with the reduced physical-state conditions (9.1) and (9.4) for the open- and closed-string sectors, these results allow us to conclude the following on inspection: Each cycle j of each sector σ of the “pure” orbifold-strings (10.1) is nothing but an ordinary untwisted 26-dimensional string with target space-time symmetry $\text{SO}(25, 1)$. This conclusion is however quite special for the “pure” orbifold-string systems with trivial H'_{26} , whereas (as seen for $H(\text{perm})_2 = \mathbb{Z}_2$ and $\mathbb{Z}_2(\text{w.s.})$ in Ref. [3]) the orbifold-string theories with non-trivial H'_{26} are generically new.

The critical-string equivalences of this section were anticipated for the “pure” orbifolds of permutation-type with $H(\text{perm})_2 = \mathbb{Z}_2$ or $\mathbb{Z}_2(\text{w.s.})$ in Ref. [3], and were verified at the interacting level [5] for the pure permutation orbifolds with $H(\text{perm})_K = \mathbb{Z}_K$, K prime. Moreover, our conclusion here was conjectured for all $H(\text{perm})_K$ in Ref. [5]. It should be added that special cases with particular non-trivial H'_{26} (see e.g. the orientation-orbifold string system in Ref. [4]) can also be equivalent to ordinary critical strings, including the critical bosonic open-closed string system. Taken together then, the orbifold-string systems of permutation-type provide several rising, ever-more twisted hierarchies of new string theories, including ordinary critical strings as the simplest cases.

We finally note that the pure permutation orbifolds

$$\frac{\text{U}(1)^{26K}}{H(\text{perm})_K}, \quad (10.11)$$

being composed entirely of ordinary closed-string sectors, exhibit *multiple gravitons*. Indeed, we have seen here that the free theories in these cases exhibit one graviton per cycle per sector. The only examples studied so far at the symmetrized, interacting level are the prime cyclic permutation orbifolds

$$\frac{\text{U}(1)^{26\lambda}}{\mathbb{Z}_\lambda}, \quad \lambda \text{ prime} \quad (10.12)$$

where the linear (diagonal) modular-invariant construction of Ref. [5] shows the total number of gravitons

$$N_\lambda = \lambda + (\lambda - 1) = 2\lambda - 1. \quad (10.13)$$

This includes in particular one graviton in each nontrivial twisted sector. The interaction (or presumably non-interaction) among these gravitons will require the construction of the twist fields (intertwiners) among the sectors, an inquiry which is beyond the scope of this paper. We similarly expect more than one graviton in the generalized permutation orbifolds with nontrivial H'_{26} , at least from the cycles of the sector corresponding to the unit elements of $H(\text{perm})_K$ and H'_{26} . On the other hand – as we will discuss in succeeding papers – the open-closed string systems of the orientation-orbifolds [3,4]

$$\frac{\text{U}(1)^{26K}}{H_-} = \frac{\text{U}(1)_L^{26} \times \text{U}(1)_R^{26}}{H_-}, \quad H_- \subset \mathbb{Z}_2(\text{w.s.}) \times H'_{26} \quad (10.14)$$

have only a single graviton for any choice of H'_{26} .

11 Conclusions

In the previous paper [6] of this series, we used BRST quantization to find the orbifold Virasoro algebras and extended physical-state conditions of the bosonic prototypes of the orbifold-string theories of permutation-type:

$$\frac{\text{U}(1)^{26K}}{H_+}, \quad \left[\frac{\text{U}(1)^{26K}}{H_+} \right]_{\text{open}}, \quad H_+ \subset H(\text{perm})_K \times H'_{26} \quad (11.1a)$$

$$\frac{\text{U}(1)^{26}}{H_-} = \frac{\text{U}(1)_L^{26} \times \text{U}(1)_R^{26}}{H_-}, \quad H_- \subset \mathbb{Z}_2(\text{w.s.}) \times H'_{26}. \quad (11.1b)$$

These theories live at cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$, where $f_j(\sigma)$ is the length of cycle j in each equivalence class σ of the permutation group $H(\text{perm})_K$ or $\mathbb{Z}_2(\text{w.s.})$. The expected sector central charges $\hat{c}(\sigma) = 26K$ of each orbifold are obtained by summing over the cycles of sector σ .

In this paper we have completed the cycle dynamics of these theories, supplementing the extended physical-state conditions of cycle j with the explicit form of the orbifold Virasoro generators as functions of the twisted matter of each cycle. Our results here are general, depending on the choice of element $\omega(\sigma) \in H'_{26}$ in the divisors of each orbifold. With these tools, we also began a systematic inquiry into the target space-time structure of these theories, including in particular the number $\hat{D}_j(\sigma)$ of target space-time dimensions in cycle j of sector σ .

We also found an equivalent, reduced description of the physical states of each cycle at reduced cycle central charge $c_j(\sigma) = 26$, emphasizing that the target space-time properties of the theories are invariant under the reduction and in fact more transparent in the reduced formulation.

As examples, the simplest cases with trivial H'_{26} (the orbifolds of “pure” permutation-type)

$$\frac{\mathrm{U}(1)^{26K}}{H(\mathrm{perm})_K}, \quad \left[\frac{\mathrm{U}(1)^{26K}}{H(\mathrm{perm})_K} \right]_{\mathrm{open}}, \quad \frac{\mathrm{U}(1)^{26}}{\mathbb{Z}_2(\mathrm{w.s.})} \quad (11.2)$$

were worked out in some detail, with the result that $\hat{D}_j(\sigma) = 26$ for each cycle j in each sector σ of these examples. Indeed, the reduced formulation transparently shows that each of these cycles is spectrally equivalent to an ordinary untwisted 26-dimensional string.

This is not the case however for the more general situation with nontrivial H'_{26} , which provides large classes of new string theories. In particular, it is clear from our discussion that the target space-time dimensionality of these theories is not generically equal to any of the central charges discussed above.

In the following paper, we will apply the general formulae developed here to study a large example of non-trivial H'_{26} , finding that the new string theories in fact exhibit many target space-times of varying dimensionality, symmetry and signature – including in particular Lorentzian target space-times with $\hat{D}_j(\sigma) \leq 26$.

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